02

In our daily life, we come across many patterns that characterise relations such as brother and sister, father and son, husband and wife. In Mathematics, there is a variety of relations, whose knowledge is crucial. Each relation has its own meaning and significance.

e.g. $\tan\theta = \dfrac{1}{\cot\theta}$ (relation between $\tan\theta$ and $\cot\theta$)

In this chapter, we will learn how to link pairs of objects from two sets and then introduce relations between the two objects in pairs.

RELATIONS AND FUNCTIONS

|TOPIC 1|

Cartesian Product

CHAPTER CHECKLIST

- Cartesian Product
- Relations
- Functions, Domain and Range of Real Functions
- Some Standard Real Functions and Their Graphs
- Algebra of Real Functions

ORDERED PAIRS

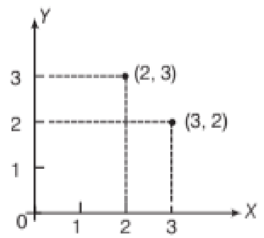
If a pair of elements written in circular bracket and grouped together in a particular order, then such a pair, is called an **ordered pair**. Ordered pair is written by listing two objects in the specific order, separated by **comma** (,) and enclosing the pair in **parentheses**.

e.g. The ordered pair of two elements a and b is denoted by (a, b) : a is first element (or first component) and b is second element (or second component).

Two ordered pairs are equal if their corresponding elements are equal.

i.e. $(a, b) = (c, d) \Rightarrow a = c$ and $b = d$

Graphically, ordered pair (a, b) represents a point in cartesian plane. So, ordered pair (2, 3) implies that **abscissa** $x = 2$ and **ordinate** $y = 3$.



It is clear from the graph that ordered pairs (2, 3) and (3, 2) represent two different points and hence they are not equal.

Note

(i) An ordered pair is not a set consisting of two elements. The order of two elements in an ordered pair is important and the two elements need not to be distinct.

(ii) $(a, b) \neq (b, a)$

EXAMPLE |1| Find the values of a and b, if

(i) $(2a-5, 4) = (5, b+6)$ (ii) $(a-3, b+7) = (3, 7)$

Sol. We know that, two ordered pairs are equal, if their corresponding elements are equal.

(i) $(2a-5, 4) = (5, b+6) \Rightarrow 2a-5 = 5$ and $4 = b+6$ [equating corresponding elements]

$\Rightarrow \quad 2a = 5+5$ and $4-6 = b$

$\Rightarrow \quad 2a = 10$ and $-2 = b$

$\Rightarrow \quad a = 5$ and $b = -2$

(ii) $(a-3, b+7) = (3, 7) \Rightarrow a-3 = 3$ and $b+7 = 7$ [equating corresponding elements]

$\Rightarrow \quad a = 3+3$ and $b = 7-7$

$\Rightarrow \quad a = 6$ and $b = 0$

CARTESIAN PRODUCT OF TWO SETS

The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$, is called the cartesian product of sets A and B and is denoted by $A \times B$.

Or

For two non-empty sets A and B, the cartesian product $A \times B$ is the set of all ordered pairs of elements from sets A and B.

In symbolic form, it can be written as

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Method to Write Cartesian Product of Two Sets

If two sets say A and B are given, then *we can write the cartesian product $A \times B$ with the help of following steps*

Step I First, write the given sets in tabular form (if not given in tabular form).

Step II Now, take an element from first set say A and form all ordered pairs with this element as first element and elements of set B as second element.

Step III Now, choose another element from first set A and form all ordered pairs with this element as first element and elements of set B as second element.

Step IV Repeat above steps for all elements of first set A and get all ordered pairs.

Now, the set of all ordered pairs, obtained in steps II, III and IV, represent the cartesian product $A \times B$.

EXAMPLE |2| Let $A = \{a, b, c\}$ and $B = \{x : x \in N, x$ is a prime number less than $5\}$. Find $A \times B$ and $B \times A$. Show that $A \times B \neq B \times A$.

Sol. Given sets are $A = \{a, b, c\}$

and $B = \{x : x \in N, x$ is a prime number less than $5\} = \{2, 3\}$

For element a of set A, all ordered pairs are $(a, 2), (a, 3)$.

For element b of set A, all ordered pairs are $(b, 2), (b, 3)$.

For element c of set A, all ordered pairs are $(c, 2), (c, 3)$.

$\therefore \quad A \times B = \{(a, 2)(a, 3), (b, 2)(b, 3)(c, 2)(c, 3)\}$

For element 2 of set B, all ordered pairs are $(2, a), (2, b), (2, c)$.

For element 3 of set B, all ordered pairs are $(3, a), (3, b), (3, c)$.

$B \times A = \{(2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$

Since, $\quad (a, 2) \neq (2, a)$

$\therefore \quad A \times B \neq B \times A$

Note

If $A \times B$ is given, then we can find $B \times A$ from it, just by reversing the position of elements of each ordered pair.

e.g. If $A \times B = \{(4, 5)(4, 6)(7, 5)(7, 6)\}$,

then $B \times A = \{(5, 4), (6, 4), (5, 7), (6, 7)\}$

METHOD TO FIND THE SETS WHEN CARTESIAN PRODUCT IS GIVEN

Sometimes the cartesian product of two sets or some elements of cartesian product of two sets are given and we have to find the sets.

For finding these two sets, we write first element of each ordered pair in first set say A and corresponding second element in second set B (say).

EXAMPLE |3| If $A \times B = \{(a, 1), (b, 3), (a, 3), (b, 1), (a, 2), (b, 2)\}$. Then, find A and B.

Sol. Here, first element of each ordered pair of $A \times B$ gives the elements of set A and corresponding second element gives the elements of set B.

$\therefore \quad A = \{a, b\}$ and $B = \{1, 3, 2\}$

Note

We write each element only one time in set, if it occurs more than one time.

Number of Elements in Cartesian Product of Two Sets

(*i*) If there are p elements in set A and q elements in set B, then there will be pq elements in $A \times B$ i.e. if $n(A) = p$ and $n(B) = q$, then

$$n(A \times B) = pq.$$

(*ii*) If A and B are non-empty sets and either A or B is an infinite set, then $A \times B$ will also be an infinite set.

(*iii*) If A or B is the null set or an empty set, then $A \times B$ will also be an empty set i.e.

$$A \times B = \phi$$

Note

If there are p elements in set A and q elements in set B, then the number of subsets of $A \times B$ is 2^{pq}

EXAMPLE |4| If the set A has 3 elements and set B has 4 elements, then find the number of elements in $A \times B$.

Sol. Given, $n(A) = 3$ and $n(B) = 4$.

$\therefore$ The number of elements in $A \times B$ is

$$n(A \times B) = n(A) \times n(B) = 3 \times 4 = 12$$

EXAMPLE |5| A and B are two sets given in such a way that $A \times B$ contains 6 elements. If three elements of $A \times B$ are (1, 3), (2, 5) and (3, 3), find A, B and remaining elements of $A \times B$.

Sol. It is given that (1, 3), (2, 5) and (3, 3) are in $A \times B$. It follows that 1, 2, 3 are elements of A and 3, 5 are elements of B.

$\therefore \quad A = \{1, 2, 3\}$ and $B = \{3, 5\}$

$\therefore \quad A \times B = \{1, 2, 3\} \times \{3, 5\}$

$\Rightarrow \quad = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)\}$

Hence, the remaining elements of $(A \times B)$ are (1, 5), (2, 3), (3, 5).

EXAMPLE |6| The cartesian product $P \times P$ has 16 elements among which are found $(a, 1)$ and $(b, 2)$. Find the set P and the remaining elements of $P \times P$.

Sol. Given, $\quad n(P \times P) = 16$

$\Rightarrow \quad n(P) \cdot n(P) = 16$

$\Rightarrow \quad n(P) = 4 \quad \text{... (i)}$

Now, as $\quad (a, 1) \in P \times P$

$\therefore \quad a \in P$ and $1 \in P$

Again, $\quad (b, 2) \in P \times P$

$\therefore \quad b \in P$ and $2 \in P$

$\Rightarrow \quad a, b, 1, 2 \in P$

From Eqs. (i), it is clear that P has exactly four elements.

$\therefore \quad P = \{a, b, 1, 2\}$

Now, $P \times P = \{a, b, 1, 2\} \times \{a, b, 1, 2\}$

$= \{(a, a), (a, b), (a, 1), (a, 2), (b, a), (b, b), (b, 1), (b, 2)$
$(1, a), (1, b), (1, 1), (1, 2), (2, a), (2, b), (2, 1), (2, 2)\}$

Hence, the remaining elements of $P \times P$ are
$(a, a), (a, b), (a, 2), (b, a), (b, b), (b, 1), (1, a), (1, b), (1, 1), (1, 2), (2, a), (2, b), (2, 1)$ and $(2, 2)$

EXAMPLE |7| Let P and Q be two sets such that $n(P) = 3$ and $n(Q) = 4$. If $(r, 4), (g, 1), (w, 3)$ and $(r, 9)$ are in $P \times Q$, then find P and Q, where r, g, w are where distinct elements. Also, write the remaining elements of $P \times Q$.

Sol. Since, r, g, w are distinct elements and $(r, 4), (g, 1), (w, 3), (r, 9)$ are elements of $P \times Q$.

$\therefore \; r, g, w \in P$ and $1, 3, 4, 9 \in Q$

$\because \quad n(P) = 3$ and $n(Q) = 4$

$\therefore \quad P = \{r, g, w\}$

and $\quad Q = \{1, 3, 4, 9\}$

Now, $P \times Q = \{r, g, w\} \times \{1, 3, 4, 9\}$

$= \{(r, 1), (r, 3), (r, 4), (r, 9), (g, 1), (g, 3), (g, 4), (g, 9), (w, 1), (w, 3), (w, 4)$ and $(w, 9)\}$

Hence, the remaining elements of $P \times Q$ are $(r, 1), (r, 3), (g, 3), (g, 4), (g, 9), (w, 1), (w, 4)$ and $(w, 9)$.

EXAMPLE |8| Let $A = \{1, 2\}$ and $B = \{3, 4\}$. How many subsets will $A \times B$ have? List them.

Sol. Given, $A = \{1, 2\}$ and $B = \{3, 4\}$

$\Rightarrow \quad n(A) = 2$ and $n(B) = 2$

$\therefore$ Total subsets of $A \times B = 2^{2 \times 2} = 2^4 = 16$

[$\because$ number of subsets $= 2^{pq}$]

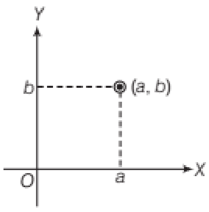
Subsets of $A \times B$ are

$\{(1, 3)\}, \{(1, 4)\}, \{(2, 3)\}, \{(2, 4)\}, \{(1, 3), (1, 4)\}, \{(1, 3), (2, 3)\}, \{(1, 3), (2, 4)\}, \{(1, 4), (2, 3)\}, \{(1, 4), (2, 4)\}, \{(2, 3), (2, 4)\}, \{(1, 3), (1, 4), (2, 3)\}, \{(1, 3), (1, 4), (2, 4)\}, \{(1, 3), (2, 3), (2, 4)\}, \{(1, 4), (2, 3), (2, 4)\}, \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ and ϕ.

Graphical Representation of Cartesian Product of Two Sets

Let A and B be two non-empty sets and let (a, b) be an ordered pair, where $a \in A$ and $b \in B$. To represent it graphically, we draw two mutually perpendicular lines OX and OY intersecting each other at O and then we represent the elements of set A on horizontal line OX and elements of set B on vertical line OY by taking a suitable scale on OX and OY.

The point of intersection of a vertical line through a and horizontal line through b represent the ordered pair (a, b).



In this manner, we mark the points corresponding to each ordered pair in $A \times B$. The set of points so obtained represents $A \times B$ graphically.

METHOD TO REPRESENT CARTESIAN PRODUCT OF TWO SETS ON GRAPH

We can represent the cartesian product of two sets on graph by representing each ordered pair on graph.

For this, *we use the following steps*

Step I Write the given sets in tabular form (if not given in tabular form).

Step II Find the cartesian product of these two sets.

Step III Draw two perpendicular lines OX and OY (say) intersecting at O and represent the elements of set A on OX and elements of set B on OY by taking a suitable scale on OX and OY.

Step IV Draw vertical dotted lines through the points representing elements of set A on OX and draw horizontal dotted lines through the points representing elements of set B on OY

Step V The point of intersections of these line (draw in step IV) represent the ordered pairs of cartesian product i.e. elements of cartesian product of given sets.

EXAMPLE |9| Let $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$. Represent the product $A \times B$ graphically.

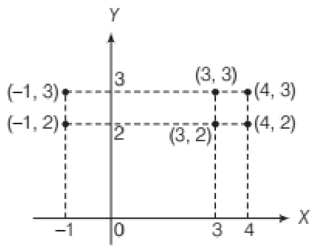
Sol. Given sets are $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$, which are in tabular form.

Now, $A \times B = \{(-1, 2), (-1, 3), (3, 2), (3, 3), (4, 2), (4, 3)\}$

Let us draw two perpendicular lines OX and OY intersecting at O. Represent the elements of set A i.e. $-1, 3, 4$ on horizontal line OX and elements of set B i.e. $2, 3$ on vertical line OY, by taking $1\text{cm} = 1\text{unit}$.

Draw vertical dotted lines through the points -1, 3 and 4 of OX and horizontal dotted lines through the points 2 and 3 of OY. Points of intersection of these lines are $(-1, 2), (-1, 3), (3, 3), (3, 2), (4, 3)$ and $(4, 2)$.

The points so obtained represent the cartesian product of sets A and B on graph.



EXAMPLE |10| If $A = \{2, 4, 6\}$ and $B = \{1, 4\}$, then show that $A \times B \neq B \times A$. Also, show it graphically.

Sol. We have, $A = \{2, 4, 6\}$ and $B = \{1, 4\}$

Then, $A \times B = \{(2, 1), (2, 4), (4, 1), (4, 4), (6, 1), (6, 4)\}$ ...(i)

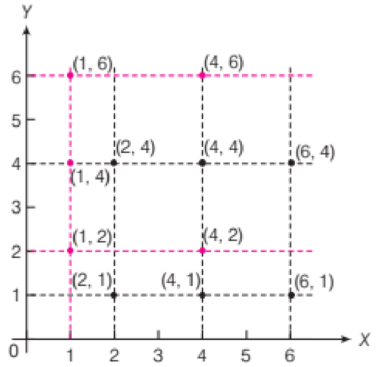
and $B \times A = \{(1, 2), (1, 4), (1, 6), (4, 2), (4, 4), (4, 6)\}$...(ii)

From Eqs. (i) and (ii), we have $A \times B \neq B \times A$

Graphical Representation Firstly, draw two perpendicular lines, say OX and OY, intersecting at O.

$\boldsymbol{A \times B}$ To show $A \times B$ graphically, we draw three dotted vertical lines through 2, 4 and 6 of OX. Also, draw two dotted horizontal lines through 1 and 4 of OY.

The point of intersection of these lines represents the ordered pairs of $A \times B$.



$\boldsymbol{B \times A}$ Now, to show $B \times A$ graphically, we draw two dotted vertical lines through 1 and 4 of OX. Also, draw three dotted horizontal lines through 2, 4 and 6 of OY.

The points of intersections of these lines represents the ordered pairs of $B \times A$.

Thus, from the graph, we can see that ordered pairs of $A \times B$ (**denoted by black dots**) have different position from the ordered pairs of $B \times A$ (**denoted by coloured dots**).

So ordered pairs of both cartesian product represent different points on graph.

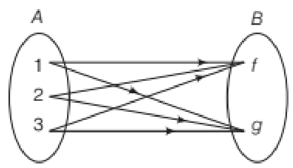
Hence, $A \times B \neq B \times A$

Diagramatic Representation of Cartesian Product of Two Sets

In order to represent $A \times B$ by an **arrow diagram**, we first draw two circles representing sets A and B one opposite to the other as shown in the given figure and write the elements of sets in the corresponding circles.

Now, we draw line segments starting from each element of set A and terminating to each element of set B.

e.g. if $A = \{1, 2, 3\}$ and $B = \{f, g\}$, then following figure gives the arrow diagram of $A \times B$.



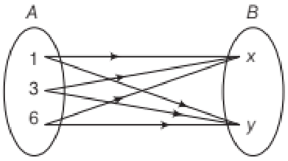
EXAMPLE |11| If $A = \{1, 3, 6\}$ and $B = \{x, y\}$, then represent the following cartesian products by an arrow diagrams

(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

Sol. We have, $A = \{1, 3, 6\}, B = \{x, y\}$

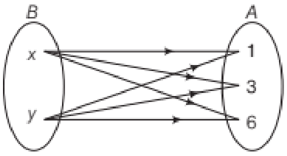
(i) $A \times B = \{1, 3, 6\} \times \{x\ y\}$
$= \{(1, x), (1, y), (3, x), (3, y), (6, x), (6, y)\}$

Required arrow diagram is



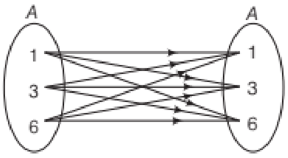
(ii) $B \times A = \{x, y\} \times \{1, 3, 6\}$
$= \{(x, 1), (x, 3), (x, 6), (y, 1), (y, 3), (y, 6)\}$

Required arrow diagram is



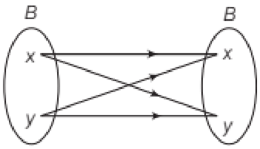
(iii) $A \times A = \{1, 3, 6\} \times \{1, 3, 6\}$
$= \{(1, 1), (1, 3), (1, 6), (3, 1), (3, 3), (3, 6), (6, 1), (6, 3) (6, 6)\}$

Required arrow diagram is



(iv) $B \times B = \{x, y\} \times \{x, y\}$
$= \{(x, x), (x, y), (y, x), (y, y)\}$

Required arrow diagram is



ORDERED TRIPLET

An ordered triplet is a list of 3 elements written in a particular order and enclosed within the circular bracket. e.g. (1, 2, 3) is an ordered triplet. Ordered triplet is also called 3-**tuple.**

Cartesian Product of Three Sets

If A, B and C are three sets, then

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

Note

In general, if $A_1, A_2, \ldots, A_n$ are n sets, then $(a_1, a_2, \ldots, a_n)$ is called a ***n*-tuple**, where $a_i \in A_i$, $i = 1, 2, \ldots, n$ and the set of all such n-tuples is called the cartesian product of $A_1, A_2, \ldots, A_n$ and is denoted by $A_1 \times A_2 \times \ldots \times A_n$.

In symbolic form,

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i, 1 \le i \le n\}$

METHOD TO WRITE THE CARTESIAN PRODUCT OF THREE SETS

Let three sets A, B and C be given, then to write the cartesian product $A \times B \times C$

we may use the following steps

***Step* I** First, write the given three sets A, B and C in tabular form (if not given in tabular form).

***Step* II** Take first two sets A and B and write the cartesian product of these two sets i.e. $A \times B$.

***Step* III** Now, consider the cartesian product obtained in step II as a set, say D, and then assume each ordered pair of $D (= A \times B)$ as element and write cartesian product of two sets D and C.

Now, the set of all ordered triplet, obtained in step III respresent the cartesian product $A \times B \times C$.

EXAMPLE |12| Find the cartesian product of three sets $A = \{1, 2\}, B = \{3, 4\}$ and $C = \{x : x \in N$ and $4 \le x \le 6\}$.

Sol. We have, $A = \{1, 2\}, B = \{3, 4\}$

and $C = \{x : x \in N$ and $4 \le x \le 6\} = \{4, 5, 6\}$

Cartesian product of A and $B = A \times B = \{1, 2\} \times \{3, 4\}$
$= \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

Let $D = A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

Now, $D \times C = A \times B \times C = \{(1, 3), (1, 4), (2, 3), (2, 4)\} \times \{4, 5, 6\}$

$= \{(1, 3, 4), (1, 3, 5), (1, 3, 6), (1, 4, 4) (1, 4, 5), (1, 4, 6)$
$(2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4, 4), (2, 4, 5), (2, 4, 6)\}$

It is the required cartesian product of three sets.

Note

$A \times B \times C = (A \times B) \times C = A \times (B \times C)$

So, we can also find the cartesian product of last two sets and then the cartesian product with first set. But position of sets will not change.

EXAMPLE |13| If $A = \{1, 2\}$, then find $A \times A \times A$.

Sol. We have, $A \times A = \{1, 2\} \times \{1, 2\}$

$$= \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

Now, $(A \times A) \times A = A \times A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\} \times \{1, 2\}$

$$= \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

SOME THEOREMS ON CARTESIAN PRODUCT OF SETS

In this section, we intend to study some results on cartesian product of sets which are given as theorems.

1. For any three non-empty sets A, B and C, we have
   (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
2. For any three non-empty sets A, B and C, we have
   $A \times (B - C) = (A \times B) - (A \times C)$
3. If A and B are any two non-empty sets, then
   $A \times B = B \times A \Leftrightarrow A = B$
4. If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$
5. If $A \subseteq B$, then
   $A \times C \subseteq B \times C$ for any non-empty set C.
6. If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$
7. For any non-empty sets A, B, C and D, we have
   $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
   Corollary
   For any non-empty sets A and B,
   $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$
   $= (A \cap B) \times (A \cap B)$
8. For any three sets A, B and C, we have
   (i) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
   (ii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$
9. Let A and B be two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.
10. Let A be a non-empty set such that $A \times B = A \times C$. Then, $B = C$

EXAMPLE BASED ON THEOREMS

EXAMPLE |14| If $A = \{1, 4\}$, $B = \{2, 3, 6\}$ and $C = \{2, 3, 7\}$, then verify that
(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Sol. We have,

$$A = \{1, 4\}, B = \{2, 3, 6\} \text{ and } C = \{2, 3, 7\}$$

(*i*) **To find $A \times (B \cup C)$**

$$B \cup C = \{2, 3, 6\} \cup \{2, 3, 7\}$$
$$\Rightarrow \quad B \cup C = \{2, 3, 6, 7\}$$
$$\therefore \quad A \times (B \cup C) = \{1, 4\} \times \{2, 3, 6, 7\}$$
$$= \{(1, 2), (1, 3), (1, 6), (1, 7), (4, 2), (4, 3), (4, 6), (4, 7)\} \quad \ldots\text{(i)}$$

To find $(A \times B) \cup (A \times C)$

$$A \times B = \{(1, 2), (1, 3), (1, 6), (4, 2), (4, 3), (4, 6)\}$$
and $$A \times C = \{(1, 2), (1, 3), (1, 7), (4, 2), (4, 3), (4, 7)\}$$
$$\therefore (A \times B) \cup (A \times C) = \{(1, 2), (1, 3), (1, 6), (1, 7), (4, 2), (4, 3), (4, 6), (4, 7)\} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

(*ii*) **To find $A \times (B \cap C)$**

$$B \cap C = \{2, 3\}$$
$$\therefore \quad A \times (B \cap C) = \{1, 4\} \times \{2, 3\}$$
$$= \{(1, 2), (1, 3), (4, 2), (4, 3)\} \quad \ldots\text{(iii)}$$

To find $(A \times B) \cap (A \times C)$

$$A \times B = \{(1, 2), (1, 3), (1, 6), (4, 2), (4, 3), (4, 6)\}$$
$$A \times C = \{(1, 2), (1, 3), (1, 7), (4, 2), (4, 3), (4, 7)\}$$
$$\therefore (A \times B) \cap (A \times C) = \{(1, 2), (1, 3), (4, 2), (4, 3)\} \quad \ldots\text{(iv)}$$

From Eqs. (iii) and (iv), we get

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 Let A and B be two sets such that $A \times B$ consists of 6 elements. If three elements of $A \times B$ are (1, 4), (2, 6) and (3, 6), then
(a) $(A \times B) = (B \times A)$
(b) $(A \times B) \neq (B \times A)$
(c) $A \times B = \{(1, 4), (1, 6), (2, 4)\}$
(d) None of the above

2 Let A and B be two non-empty sets having n elements in common. Then, the number of elements common to $A \times B$ and $B \times A$ is
(a) $2n$ (b) n
(c) n^2 (d) None of these

3 If $n(A \times B) = 45$, then $n(A)$ cannot be
(a) 15 (b) 17
(c) 5 (d) 9

4 If $A = \{x : x^2 - 5x + 6 = 0\}$; $B = \{2, 4\}$, $C = \{4, 5\}$, then $A \times (B \cap C)$ is

(a) $\{(2, 4), (3, 4)\}$ (b) $\{(4, 2), (4, 3)\}$
(c) $\{(2, 4), (3, 4), (4, 4)\}$ (d) $\{(2, 2), (3, 3), (4, 4), (5, 5)\}$

5 If $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, then $(A \times B) \cap (B \times C)$ is equal to

(a) $\{(1, 4)\}$ (b) $\{(3, 4)\}$
(c) $\{(1, 4), (3, 4)\}$ (d) None of these

VERY SHORT ANSWER Type Questions

6 Find x and y, if $(x + 6, y - 2) = (0, 6)$.

7 Find x and y, if $(x + 2, 4) = (5, 2x + y)$.

8 If $A = \{a, b, c\}$ and $B = \{r, s\}$, then find
(i) $A \times B$ (ii) $B \times A$

9 If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$, then find A and B.

10 If $A = \{1, 3, 5\}$ and $B = \{2, 3\}$ then show that $A \times B \neq B \times A$.

11 If $A = \{1, 2, 3\}$ and $B = \{5, 6\}$, then show that $A \times B \neq B \times A$.

12 If $A = \{a, b\}$, $B = \{c, d\}$ and $C = \{d, c, e\}$, then find $A \times \{B \cup C\}$.

13 The cartesian product $A \times A$ has 9 elements among which are found $(-1, 0)$ and $(0, 1)$. Find the set A and the remaining element of $A \times A$.

14 If $P = \{1, 4\}$, then find $P \times P \times P$.

SHORT ANSWER Type Questions

15 If $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$, then verify that $(A \times C) \subset (B \times D)$.

16 Let A and B be two sets such that $n(A) = 5$ and $n(B) = 2$. If a, b, c, d, e are distinct and $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are elements of $A \times B$, find A and B.

LONG ANSWER Type Questions

17 Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$. Represent following product graphically.
(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$

18 If $A = \{a, b, c\}$ and $B = \{x, y\}$, then represent the following cartesian product by an arrow diagram.
(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

19 If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$, then find
(i) $A \times (B \cap C)$ (ii) $(A \times B) \cap (A \times C)$

20 If $A = \{a, d\}$, $B = \{b, c, e\}$ and $C = \{b, c, f\}$, then verify that
(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

HINTS & ANSWERS

1. (b) Since, (1, 4), (2, 6) and (3, 6) are elements of $A \times B$, it follows that 1, 2, 3 are elements of A and 4, 6 are elements of B. It is given that $A \times B$ has 6 elements.
So, $A = \{1, 2, 3\}$ and $B = \{4, 6\}$
Hence, $A \times B = \{1, 2, 3\} \times \{4, 6\}$
$= \{(1, 4), (1, 6), (2, 4), (2, 6), (3, 4), (3, 6)\}$
and $B \times A = \{4, 6\} \times \{1, 2, 3\}$
$= \{(4, 1), (4, 2), (4, 3), (6, 1), (6, 2), (6, 3)\}$

2. (c) We know that,
$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
$\therefore$ $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$
$\Rightarrow$ $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$
It is given that $A \cap B$ has n elements
$\therefore$ $(A \cap B) \times (A \cap B)$ has n^2 elements.
Thus, $(A \times B) \cap (B \times A)$ has n^2 elements in common.

3. (b) We have, $n(A \times B) = 45$
$\Rightarrow$ $n(A) \times n(B) = 45$
$\Rightarrow$ $n(A)$ and $n(B)$ are factors of 45 such that their product is 45.
Here, $n(A)$ cannot be 17.

4. (a) We have, $A = \{2, 3\}$, $B = \{2, 4\}$ and $C = \{4, 5\}$
$\therefore$ $B \cap C = \{4\}$
$\Rightarrow$ $A \times (B \cap C) = \{2, 3\} \times \{4\}$
$= \{(2, 4), (3, 4)\}$

5. (b) $A \times B = \{1, 2, 3\} \times \{3, 4\}$
$= \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$
and $B \times C = \{3, 4\} \times \{4, 5, 6\}$
$= \{(3, 4), (3, 5), (3, 6), (4, 4), (4, 5), (4, 6)\}$
$\therefore (A \times B) \cap (B \times C) = \{(3, 4)\}$

6. $x = -6$, $y = 8$

7. $x = 3$, $y = -2$

8. (i) $A \times B = \{(a, r), (a, s), (b, r), (b, s), (c, r), (c, s)\}$
(ii) $B \times A = \{(r, a), (r, b), (r, c), (s, a), (s, b), (s, c)\}$

9. $A = \{a, b\}$, $B = \{x, y\}$

10. Solve as Example 2.

11. Solve as Example 2.

12. $B \cup C = \{c, d, e\}$
$A \times (B \cup C) = \{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$

13. $(-1, 0) \in A \times A$, therefore $-1, 0 \in A$ and $(0, 1) \in A \times A$, therefore $0, 1 \in A$

$\Rightarrow \quad -1, 0, 1 \in A$

$\Rightarrow A = \{-1, 0, 1\}$ and remaining elements of $A \times A$ are $(-1, -1), (-1, 1), (0, -1), (0, 0), (1, -1), (1, 0), (1, 1)$.

14. $P \times P \times P = \{(1, 1, 1), (1, 1, 4), (1, 4, 1), (1, 4, 4), (4, 1, 1), (4, 1, 4), (4, 4, 1), (4, 4, 4)\}$

15. $A \times C = \{(1,5), (1, 6), (2, 5), (2, 6), (3,5), (3,6)\}$

$B \times D = \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}$

Here, elements $(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6) \in A \times C$ are also belongs to $B \times D$, therefore $(A \times C) \subset (B \times D)$.

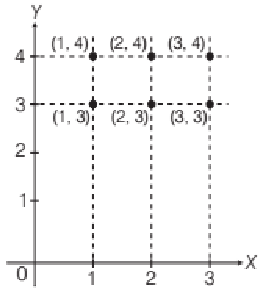
16. $\because (a, 2), (b, 3), (c, 2), (d, 3), (e, 2) \in A \times B$

$\therefore \quad a, b, c, d, e \in A$ and $2, 3 \in B$

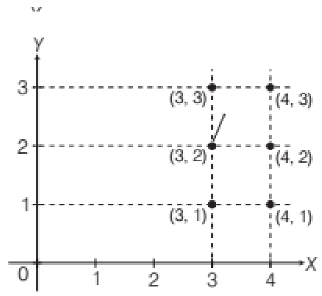
Also, it is given that $n(A) = 5$, $n(B) = 2$

$\therefore \quad A = \{a, b, c, d, e\}$, $B = \{2, 3\}$

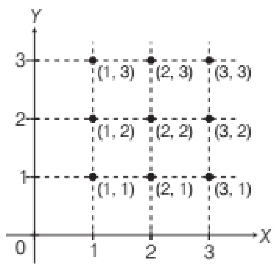
17. (i)



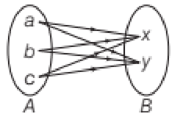
(ii)



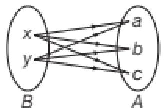
(iii)



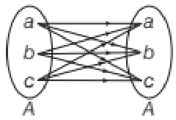
18. (i)



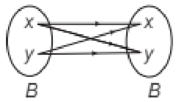
(ii)



(iii)



(iv)



19. (i) $(B \cap C) = \{4\}$

Now, $A \times (B \cap C) = \{1, 2, 3\} \times \{4\}$

$= \{(1, 4), (2, 4), (3, 4)\}$

(ii) $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$

$A \times C = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

$\therefore \quad (A \times B) \cap (A \times C) = \{(1, 4), (2, 4), (3, 4)\}$

20. (i) **To determine $A \times (B \cup C)$**

$B \cup C = \{b, c, e\} \cup \{b, c, f\}$

$= \{b, c, e, f\}$

$\therefore \quad A \times (B \cup C) = \{a, d\} \times \{b, c, e, f\}$

$= \{(a, b), (a, c), (a, e), (a, f), (d, b), (d, c), (d, e), (d, f)\}$ ...(i)

To determine $(A \times B) \cup (A \times C)$

$A \times B = \{a, d\} \times \{b, c, e\}$

$= \{(a, b), (a, c), (a, e), (d, b), (d, c), (d, e)\}$

$A \times C = \{a, d\} \times \{b, c, f\}$

$= \{(a, b), (a, c), (a, f), (d, b), (d, c), (d, f)\}$

$\therefore \quad (A \times B) \cup (A \times C) = \{(a, b), (a, c), (a, e), (a, f), (d, b), (d, c), (d, e), (d, f)\}$ ...(ii)

From Eqs. (i) and (ii), we get

$A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $B \cap C = \{b, c\}$;

$A \times (B \cap C) = \{(a, b), (a, c), (d, b), (d, c)\}$

and $(A \times B) \cap (A \times C) = \{(a, b), (a, c), (d, b), (d, c)\}$

$\Rightarrow \quad A \times (B \cap C) = (A \times B) \cap (A \times C)$

|TOPIC 2|

Relations

Let A and B be two non-empty sets. Then, a relation R from set A to set B is a subset of cartesian product $A \times B$ i.e. $R \subseteq A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called **image** of first element.

If $(a, b) \in R$, then we write it as $a\,R\,b$ and *we say that* **a is related to b by the relation R.**

If $(a, b) \notin R$, then we write it as $a\not R b$ and *we say that* **a is not related to b by the relation R.**

EXAMPLE |1| If $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$, then which of the following are relations from A to B? Give answer.

(i) $R_1 = \{(1, 5), (2, 7), (3, 8)\}$

(ii) $R_2 = \{(6, 2), (3, 7), (4, 7)\}$

Every relation from A to B is a subset of $A \times B$.

Sol. We have, $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$

$A \times B = \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}$

(i) Since, $R_1 \subseteq A \times B$, therefore R_1 is a relation from A to B.

(ii) Since, $(6, 2) \in R_2$ but $(6, 2) \notin A \times B$, therefore $R_2 \not\subseteq A \times B$. Thus, R_2 is not a relation from A to B.

Representation of a Relation

A relation can be represented algebraically by roster form or by set-builder form and visually it can be represented by an arrow diagram *which are given below*

(i) **Roster form** In this form, we represent the relation by the set of all ordered pairs belongs to R.

e.g. Let R is a relation from set $A = \{-3, -2, -1, 1, 2, 3\}$ to set $B = \{1, 4, 9, 10\}$, defined by $aRb \Leftrightarrow a^2 = b$,

Then, $(-3)^2 = 9, (-2)^2 = 4, (-1)^2 = 1, (2)^2 = 4, (3)^2 = 9.$

Then, in roster form, R can be written as

$$R = \{(-1, 1), (-2, 4), (1, 1), (2, 4), (-3, 9), (3, 9)\}$$

(ii) **Set-builder form** In this form, we represent the relation R from set A to set B as

$R = \{(a, b): a \in A, b \in B$ and the rule which relate the elements of A and $B\}$

e.g. Let R is a relation from set $A = \{1, 2, 4, 5\}$ to set

$$B = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}\right\} \text{ such that}$$

$$R = \left\{(1, 1), \left(2, \frac{1}{2}\right)\left(4, \frac{1}{4}\right)\left(5, \frac{1}{5}\right)\right\}$$

Then, in set-builder form, R can be written as

$$R = \left\{(a, b) : a \in A, b \in B \text{ and } b = \frac{1}{a}\right\}$$

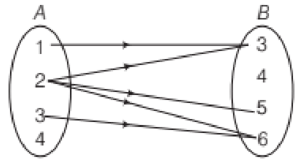
Note

We cannot write every relation from set A to set B in set-builder form.

(iii) **Arrow diagram** To represent a relation by an arrow diagram, we draw arrows from first element to second element of all ordered pairs belonging to relation R.

e.g. Let $R = \{(1, 3) (2, 5) (3, 6) (2, 6) (2, 3)\}$ be a relation from set $A = \{1, 2, 3, 4\}$ to set $B = \{3, 4, 5, 6\}$.

Then, by an arrow diagram, it can be represented as



Note

(i) If $R = \phi$, then R is called an ***empty relation.***

(ii) If $R = A \times B$, then R is called the ***universal relation.***

(iii) If R_1 and R_2 are two relations from A to B, then $R_1 \cup R_2$, $R_1 \cap R_2$ and $R_1 - R_2$ are also relations from A to B.

EXAMPLE |2| Express $R = \{(a, b): 2a + b = 5; a, b \in W\}$ as the set of ordered pairs (in roster form).

Sol. Given, $R = \{(a, b): 2a + b = 5; a, b \in W\}$

Here, W represent set of whole numbers.

When $a = 0, b = 5$

When $a = 1, b = 3$

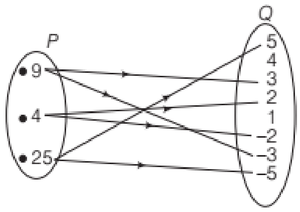
When $a = 2, b = 1$

For $a \geq 3$, the value of b given by the above relation are not whole numbers.

$\therefore \quad R = \{(0, 5), (1, 3), (2, 1)\}$

EXAMPLE |3| Write the relation between sets P and Q given by an arrow diagram in

(i) roster form (ii) set-builder form



Sol. From arrow diagram, we have

$P = \{9, 4, 25\}$ and $Q = \{5, 4, 3, 2, 1, -2, -3, -5\}$

Here, the relation R is 'x is the square of y',
where $x \in P$ and $y \in Q$.

In roster form, it can be written as

$R = \{(9, 3)(9, -3)(4, 2)(4, -2)(25, 5)(25, -5)\}$

In set-builder form, R can be written as

$R = \{(x, y) : x \in P, y \in Q$ and x is the square of $y\}$

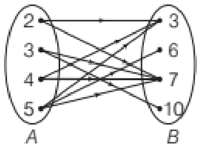
EXAMPLE |4| A relation R is defined, from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$, as $(x, y) \in R \Leftrightarrow x$ is relatively prime to y. Express R as a set of ordered pairs and depict this relations by arrow diagram.

Sol. Given, $R = \{(x, y): x \in A, y \in B, x$ is relatively prime to $y\}$

Here, x is relatively prime to y means $\gcd(x, y) = 1$.

$\therefore$ $R = \{(2, 3), (2, 7), (3, 7), (3, 10), (4, 3), (4, 7), (5, 3), (5, 6), (5, 7)\}$

Visual representation is shown below:



> ### *Number of Relations*
>
> Let A and B be any two non-empty finite sets containing m and n elements, respectively.
>
> Then, number of ordered pairs in $A \times B$ is mn.
>
> $\therefore$ Total number of subsets of $A \times B$ is 2^{mn}.
>
> Hence, the total number of relations from set A to B is 2^{mn}, because each relation from A to B is a subset of $A \times B$.
>
> Among these 2^{mn} relation, the void relation ϕ and the universal relation $A \times B$ are **trivial relations** from A to B.

EXAMPLE |5| If $A = \{a, b\}$ and $B = \{2, 3\}$, then find the number of relations from A to B.

Number of relations from A to $B = 2^{n(A) \times n(B)} = 2^{n(A \times B)}$.

Sol. We have, $A = \{a, b\}$ and $B = \{2, 3\}$,

$\therefore n(A \times B) = n(A) \times n(B) = 2 \times 2 = 4$

Now, number of subsets of $A \times B = 2^{n(A \times B)} = 2^4 = 16$

Thus, the number of relations from A to B is 16.

EXAMPLE |6| If $n(A) = 3$ and $B = \{2, 3, 4, 6, 7, 8\}$, then find the number of relations from A to B.

Sol. Given, $n(A) = 3$

and $B = \{2, 3, 4, 6, 7, 8\} \Rightarrow n(B) = 6$

$\therefore$ Number of relations from A to $B = 2^{n(A) \times n(B)}$
$= 2^{3 \times 6} = 2^{18}$

DOMAIN, CODOMAIN AND RANGE OF RELATION

Let us consider the relation R from A to B. Then, the set of first elements of all the ordered pairs present in R is called the **domain** of relation and the set of second elements of all the ordered pairs present in R is called **range** of the relation. The set B is called the **codomain** of relation R

i.e. **Domain** $(R) = \{a : (a, b) \in R\}$

and **Range** $(R) = \{b : (a, b) \in R\}$

Note that the domain of a relation from set A to B is a subset of set A and range is a subset of set B i.e. range is a subset of codomain. (Range $\subseteq$ Codomain)

e.g. Let $R = \{(1, 2), (1, 5), (3, 4), (6, 7)\}$ be a relation from set $A = \{1, 3, 6, 7\}$ to set $B = \{2, 4, 5, 6, 7\}$.

Then, domain $= \{1, 3, 6\}$, range $= \{2, 4, 5, 7\}$

and codomain $=$ set $B = \{2, 4, 5, 6, 7\}$

Clearly, Range $\subseteq$ Codomain

EXAMPLE |7| A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows $(x, y) \in R \Leftrightarrow x$ divides y. Express R as a set of ordered pairs and determine the domain and range of R.

Sol. We have, $A = \{2, 3, 4, 5\}$ and $B = \{3, 6, 7, 10\}$

Relation R is defined as $(x, y) \in R \Leftrightarrow x$ divides y.

$\therefore$ 2 divides 6, 2 divides 10, 3 divides 3, 3 divides 6, 5 divides 10.

$\therefore$ $R = \{(2, 6)(2, 10)(3, 3)(3, 6)(5, 10)\}$

$\because$ Domain $(R) =$ Set of first elements of the ordered pairs present in R

$\therefore$ Domain of $R = \{2, 3, 5\}$

$\because$ Range $(R) =$ Set of second elements of the ordered pairs present in R

$\therefore$ Range of $R = \{3, 6, 10\}$

EXAMPLE |8| Determine the domain and range of the relation $R = \{(x, y) : x \in N, y \in N$ and $x + y = 10\}$.

Sol. We have, $R = \{(x, y) : x \in N, y \in N$ and $x + y = 10\}$

$\Rightarrow$ $R = \{(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)\}$

$\therefore$ Domain $(R) =$ Set of first elements of the ordered pairs present in R

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Range $(R) =$ Set of second elements of the ordered pairs present in $R = \{9, 8, 7, 6, 5, 4, 3, 2, 1\}$

EXAMPLE |9| Let $A = \{1, 2, 3, 4, 5\}$. Define a relation R from A to A by $R = \{(x, y) : y = x - 1, x, y \in A\}$

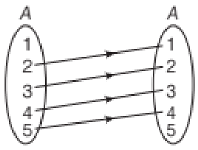
(i) Depict this relation using an arrow diagram.

(ii) Write down the domain, codomain and range of R.

Sol. (i) Given, $R = \{(x, y) : y = x - 1, x, y \in A\}$

$\therefore \quad R = \{(2, 1), (3, 2), (4, 3), (5, 4)\}$

The corresponding arrow diagram is shown in following figure.



(ii) It is clear from the arrow diagram, that domain = {2, 3, 4, 5}, codomain = {1, 2, 3, 4, 5} and range = {1, 2, 3, 4}.

EXAMPLE |10| Determine the domain and range of the relation $R = \{(a, b) : a \in N, a < 5, b = 4\}$

Sol. Given, $R = \{(a, b) : a \in N, a < 5, b = 4\}$

$\Rightarrow \quad R = \{(1, 4), (2, 4), (3, 4), (4, 4)\}$

Domain $(R) = \{a : (a, b) \in R\} = \{1, 2, 3, 4\}$

Range $(R) = \{b : (a, b) \in R\} = \{4\}$

Inverse Relation

Let $R \subseteq A \times B$ be a relation from a set A to a set B. Then, the inverse relation of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

Thus, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}, \forall\, a \in A, b \in B$

Clearly, domain (R^{-1}) = range (R)

and range (R^{-1}) = domain (R)

EXAMPLE |11| Find the inverse relation (R^{-1}) in each of the following cases

(i) $R = \{(1, 2), (1, 3), (2, 3), (3, 2), (5, 6)\}$

(ii) $R = \{(x, y) : x, y \in N, x + 2y = 8\}$

(iii) R is a relation from {11, 12, 13} to {8, 10, 12} defined by $y = x - 3$

Sol. (i) Given, $R = \{(1, 2), (1, 3), (2, 3), (3, 2), (5, 6)\}$

$\Rightarrow R^{-1} = \{(2, 1), (3, 1), (3, 2), (2, 3), (6, 5)\}$

(ii) $R = \{(x, y) : x, y \in N, x + 2y = 8\}$

$\because x + 2y = 8$

When $x = 2, y = 3$

When $x = 4, y = 2$

When $x = 6, y = 1$

$\Rightarrow \quad R = \{(2, 3), (4, 2), (6, 1)\}$

$\Rightarrow \quad R^{-1} = \{(3, 2), (2, 4), (1, 6)\}$

(iii) Here, $R = \{(x, y) : x \in \{11, 12, 13\}, y \in \{8, 10, 12\}$ and $y = \{x - 3\}$

$\because \quad y = x - 3$

When $x = 11, y = 8$

When $x = 13, y = 10$

$\therefore \quad R = \{(11, 8), (13, 10)\}$

$\Rightarrow \quad R^{-1} = \{(8, 11), (10, 13)\}$

EXAMPLE |12| Let A be the set of first 10 natural numbers and let R be a relation on A defined by

$(x, y) \in R \Leftrightarrow x + 2y = 10,$

i.e $R = \{(x, y) : x \in A, y \in A$ and $x + 2y = 10\}$.

Express R and R^{-1} as sets of ordered pair. Also, determine

(i) domain of R and R^{-1} (ii) Range of R and R^{-1}.

Sol. Given, $R = \{(x, y) : x \in A, y \in A$ and $x + 2y = 10\}$

Here, $x + 2y = 10$

$\Rightarrow \quad y = \dfrac{10 - x}{2} \forall\, x, y \in A$ ...(i)

and $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

From Eq. (i),

When $x = 2, y = 4$; When $x = 4, y = 3$

When $x = 6, y = 2$; When $x = 8, y = 1$

$\therefore \quad R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

$\Rightarrow \quad R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, domain $(R) = \{2, 4, 6, 8\}$ = Range (R^{-1})

and range $(R) = \{4, 3, 2, 1\}$ = Domain (R^{-1})

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 Two finite sets A and B have m and n elements respectively. If the total number of relation from A to B is 64, then the possible values of m and n can be

(a) 1 and 5 (b) 2 and 4

(c) 2 and 3 (d) 1 and 4

2 The relation on the set $A = \{x : |x| < 3, x \in Z\}$ is defined by $R = \{(x, y) : y = |x|, x \neq -1\}$. Then, the number of elements in the power set of R is

(a) 32 (b) 16

(c) 8 (d) 64

3 Let $n(A) = m$ and $n(B) = n$. Then, the total number of non-empty relations that can be defined from A to B is **[NCERT Exemplar]**

(a) m^n (b) $n^m - 1$

(c) $mn - 1$ (d) $2^{mn} - 1$

4 The relation R defined on the set of natural numbers as $\{(a, b) : a$ differs from b by 3$\}$, is given by

(a) $\{(1, 4), (2, 5), (3, 6), \ldots\}$

(b) $\{(4, 1), (5, 2), (6, 3), \ldots\}$

(c) $\{(1, 3), (2, 6), (3, 9), \ldots\}$

(d) None of the above

5 If a relation R is defined on the set Z of integers as follows $(a, b) \in R \Leftrightarrow a^2 + b^2 = 25$, then domain (R) is equal to

(a) $\{3, 4, 5\}$ (b) $\{0, 3, 4, 5\}$

(c) $\{0, \pm 3, \pm 4, \pm 5\}$ (d) None of these

VERY SHORT ANSWER Type Questions

6 If $A = \{x, y, z\}$ and $B = \{1, 2\}$, then find the number of relation from A to B. **[NCERT]**

7 Determine the domain and range of the relation R, defined by $R = \{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}$. **[NCERT]**

8 If $R = \{(x, y) : x, y \in N, x < 5$ and $y = 3\}$ is a relation, then find domain and range of R.

9 Let $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$. If a relation R from A to B is defined by $R = \{(x, y)$: the difference between x and y is odd, $x \in A, y \in B\}$, then write R in roster form. **[NCERT]**

10 If $A = \{1, 2, 3, 4, 6\}$ and R be the relation defined on A by $R = \{(a, b) : a \in A, b \in A$ and a divides $b\}$, then find domain and range of R.

11 Let $A = \{1, 2, 3\}$; $B = \{a, b, c, d\}$ be two sets and $R = \{(1, a), (1, c), (2, d), (2, c)\}$ be a relation from A to B, then find R^{-1}, range of R and domain of R.

SHORT ANSWER Type Questions

12 Determine the domain and range of the relations R, where $R = (x, x^3) : x$ is a prime number less than 10$\}$ **[NCERT]**

13 Express $R = \{(x, y) : x^2 + y^2 = 25$, where $x, y \in W\}$ as a set of ordered pairs

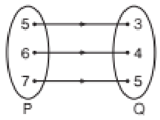
14 Determine the domain and range of the relation R, where

(i) $R = \{(-3, 1), (-1, 1), (1, 0), (3, 0)\}$

(ii) $R = \{(x - 2, x^2) : x$ is a prime number less than 15$\}$

15 Let $A = \{2, 4, 6, 9\}$ and $B = \{4, 6, 18, 27, 54\}$, then find the set of ordered pair (a, b) such that $a \in A$, $b \in B$, a is factor of b, and $a < b$.

16 The given figure shows a relationship between the sets P and Q. Write this relation



(i) in set- builder form (ii) in roster form

(iii) What is its domain and range?

17 Suppose a relation $R = \{(1, 4), (2, 5), (4, 4), (6, 5)\}$ is defined from A to B. Find the inverse of R. Also, find the domain and range of R^{-1}?

LONG ANSWER Type Questions

18 Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16, 25\}$ and R be a relation defined from A to B, as $R = \{(x, y) : x \in A, y \in B$ and $y = x^2\}$

(i) Depict this relation using arrow diagram

(ii) Find domain of R.

(iii) Find range of R.

(iv) Write codomain of R

19 If $A = \{1, 2, 3, 4, 5, 6\}$, then define a relation R from A to A by $R = \{(x, y) : y = x + 1\}$

(i) Depict this relation using an arrow diagram.

(ii) Write down the domain, co-domain and range of R.

20 Suppose a set $A = \{$January, February, August$\}$ and set $B = \{28, 15, 30\}$. Write a relation R given by $R = \{(a, b) \in A \times B$, where a is month and b is date$\}$. Also, find R^{-1}.

HINTS & ANSWERS

1. (c) Clearly, $n(A \times B) = n(A) \times n(B) = mn$

$\therefore$ Total number of relation from A to $B = 2^{mn}$

Thus, we have $2^{mn} = 64 = 2^6$

$\Rightarrow \quad mn = 6$

Hence, possible value of m and n are 2 and 3, respectively.

2. (b) We have, $A = \{x : |x| < 3, x \in Z\}$

$= \{-2, -1, 0, 1, 2\}$

and $R = \{(x, y) : y = |x|, x \neq -1\}$

$= \{(-2, 2), (0, 0), (1, 1), (2, 2)\}$

$\therefore \quad n(P(R)) = 2^{n(R)} = 2^4 = 16$

3. (d) Given, $n(A) = m$ and $n(B) = n$

$\therefore$ Total number of relations from A to $B = 2^{mn}$

$\therefore$ Total number of non-empty relations from A to $B = 2^{mn} - 1$

4. (b) Given, $R = \{(a, b) : a - b = 3\} = \{(4, 1), (5, 2), (6, 3), \ldots\}$

5. (c) We have, $(a, b) \in R \Leftrightarrow a^2 + b^2 = 25$

$\Rightarrow \quad b = \pm\sqrt{25 - a^2}$

Clearly, $a = 0 \Rightarrow b = \pm 5$

$a = \pm 3 \Rightarrow b = \pm 4$

$a = \pm 4 \Rightarrow b = \pm 3$

and $a = \pm 5 \Rightarrow b = 0$

Hence, domain $(R) = \{0, \pm 3, \pm 4, \pm 5\}$.

6. 64

7. $R = \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9), (5, 10)\}$

Domain $(R) = \{0, 1, 2, 3, 4, 5\}$; Range $(R) = \{5, 6, 7, 8, 9, 10\}$

8. $R = \{(1, 3), (2, 3), (3, 3), (4, 3)\}$

$\therefore$ Domain $(R) = \{1, 2, 3, 4\}$ and range $(R) = \{3\}$

9. $R = \{(x, y)$: the difference between x and y is odd $x \in A, y \in B\}$

In roster form

[illegible]

10. Let $R = \{(a, b) : a \in A, b \in A$, and a divides $b\}$

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$

Domain $= \{1, 2, 3, 4, 6\} = A$

Range $= \{1, 2, 3, 4, 6\} = A$

11. $R = \{(1, a), (1, c), (2, d)(2, c)\} \Rightarrow R^{-1} = \{(a, 1), (c, 1), (d, 2), (c, 2)\}$

Domain $(R) = \{1, 2\}$; Range $(R) = \{a, c, d\}$

12. $R = (x, x^3) : x$ is a prime number less than 10$\}$

$R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

[$\because$ 2, 3, 5, 7 are primes less than 10]

Domain $(R) = \{2, 3, 5, 7\}$

Range $(R) = \{8, 27, 125, 343\}$

13. Here, W is the set of whole number

$\because \quad x^2 + y^2 = 25 \quad \ldots$(i)

Put $x = 0, 1, 2, 3, 4, 5$ in Eq. (i) and obtain the value of y.

Hence, $R = \{(0, 5), (3, 4), (4, 3), (5, 0)\}$

14. (i) Given, $R = \{(-3, 1), (-1, 1), (1, 0), (3, 0)\}$

Domain $(R) = \{-3, -1, 1, 3\}$; Range $(R) = \{1, 0\}$

(ii) Given, $R = \{(x - 2, x^2) : x$ is a prime number less than 15$\}$

$\therefore R = \{(0, 4), (1, 9), (3, 25), (5, 49), (9, 121), (11, 169)\}$

Domain $(R) = \{0, 1, 3, 5, 9, 11\}$

Range $(R) = \{4, 9, 25, 49, 121, 169\}$

15. Here, $A = \{2, 4, 6, 9\}$; $B = \{4, 6, 18, 27, 54\}$

Let $R = \{(a, b) : a \in A, b \in B, a$ is factor of b and $a < b\}$

Then, $R = \{(2, 4), (2, 6), (2, 18), (2, 54), (6, 18), (6, 54), (9, 18) (9, 27), (9, 54)\}$

16. (i) $\{(x, y) : x \in P, y \in Q, x = y + 2\}$

(ii) $\{(5, 3), (6, 4), (7, 5)\}$

(iii) Domain $(R) = \{5, 6, 7\}$ and

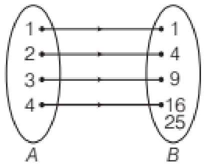
Range $(R) = \{3, 4, 5\}$

17. Given, $R = \{(1, 4), (2, 5), (4, 4), (6, 5)\}$

$\Rightarrow \quad R^{-1} = \{(4, 1), (5, 2), (4, 4), (5, 6)\}$

Domain $(R^{-1}) = \{4, 5\}$ and Range $(R^{-1}) = \{1, 2, 4, 6\}$

18. (i) The relation $R = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$



(ii) Domain $(R) = \{1, 2, 3, 4\}$

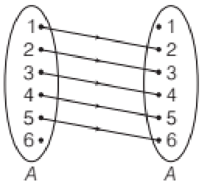
(iii) Range $(R) = \{1, 4, 9, 16\}$

(iv) Codomain $(R) = \{1, 4, 9, 16, 25\}$

19. Given, $A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(x, y) : y = x + 1\} = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

(i) The arrow diagram is shown below



(ii) Clearly, domain $(R) = \{1, 2, 3, 4, 5\}$

Range $(R) = \{2, 3, 4, 5, 6\}$;

Codomain $= \{1, 2, 3, 4, 5, 6\}$

20. Given, $R = \{(a, b) \in A \times B$, where a is month and b is date$\}$

$\therefore R = \{$(January, 28), (January, 15), (January, 30), (February, 28), (February, 15), (August, 28), (August, 15), (August, 30)$\}$

$\Rightarrow R^{-1} = \{$(28, January), (15, January), (30, January), (28, February), (15, February), (28, August), (15, August), (30, August)$\}$

|TOPIC 3|
Functions, Domain and Range of Real Function

A special type of relation is called **function.** The word 'function' is derived from a Latin word meaning **operation**. Function is a very important concept in Mathematics and it shows the correspondence between two sets. A function is also known as **mapping** or **map.**

FUNCTION AS A RELATION

A relation f from a non-empty set A to a non-empty set B is said to be a function, if every element of set A has one and only one image in set B.

In other words, we can say that a function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element or component.

If f is a function from a set A to a set B, then we write

$$f : A \rightarrow B \text{ or } A \xrightarrow{f} B$$

and it is read as f is a function from A to B or f maps A to B.

If $(a, b) \in f$, then we can also write it as $f(a) = b$. Here, b is called the **image** of a under f and a is called the **pre-image** of b under f.

e.g. Let $A = \{2, 3, 4\}$, $B = \{3, 4, 5\}$ and f_1, f_2 and f_3 be three subsets of $A \times B$ defined as

$$f_1 = \{(2, 3)\ (3, 4)\ (4, 5)\}$$
$$f_2 = \{(2, 3)\ (2, 4)\ (3, 4), (4, 5)\}$$
and $$f_3 = \{(2, 4)\ (3, 5)\}$$

Then, f_1 is a function from A to B, since every element of A has one and only one image in set B.

f_2 is not a function from A to B, since $2 \in A$ has two images 3 and 4 in B.

Also, f_3 is not a function from A to B, since $4 \in A$ has no image in B.

EXAMPLE |1| Which of the following relations are functions?

(i) {(3, 3), (4, 2), (5, 1), (6, 0), (7, 7)}
(ii) {(2,0), (4, 8), (2, 1), (3, 6)}

If first element of each ordered pair is different with other, then given relation is a function.

Sol. (i) {(3, 3), (4, 2), (5, 1), (6, 0), (7, 7)}

It is a function, because first element of each ordered pair is different.

(ii) {(2, 0), (4, 8), (2, 1), (3, 6)}

It is not a function, because first elements of (2, 0) and (2, 1) are same.

Function as a Correspondence

A function f from a non-empty set A to a non-empty set B is a **rule or method or correspondence** which associates elements of set A to elements of set B such that

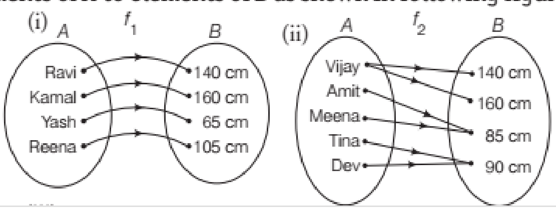
(i) All elements of set A are associated to elements in set B.
(ii) An element of set A is associated with one and only one element of set B.

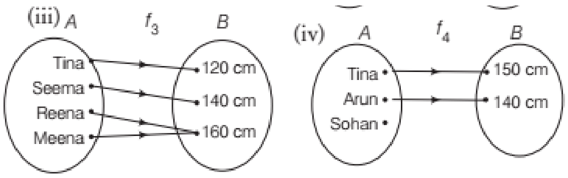
In other words, **a function 'f' from a non-empty set A to a non-empty set B, associate each element of set A to a unique element of set B.**

CHARACTERISTICS OF A FUNCTION $f: A \rightarrow B$

(i) For each element $x \in A$, there is unique element $y \in B$.
(ii) The element $y \in B$ is called the **image** of x under the function f. Also, y is called the **value of function f at x** i.e. $f(x) = y$.
(iii) $f : A \rightarrow B$ is not a function if there is an element in A which has more than one image in B. But more than one element of A may be associated to the same element of B.
(iv) $f : A \rightarrow B$ is not a function, if an element in A does not have an image in B.

EXAMPLE |2| Let A be the set of persons and B be the set of their heights and let f_1, f_2, f_3 and f_4 be rules associating elements of A to elements of B as shown in following figures.





Check whether f_1, f_2, f_3 and f_4 are functions or not and why?

Sol. In f_1 [arrow diagram (i)], we observe that each element of A is associated with exactly one element of B, Therefore, f_1 is a function from A to B.

In f_2 [arrow diagram (ii)], we observe that Vijay is associated with 140 cm and 160 cm. Any person can have only one unique height, so the height of Vijay cannot be 140 cm and 160 cm together. Hence, $f_2 : A \rightarrow B$ is not a function. In f_3 [arrow diagram (iii)], we observe that Reena and Meena are associated with 160 cm. It is possible that the height of two persons may be equal. Hence, f_3 is a function from A to B.

In f_4 [arrow diagram (iv)], we observe that Sohan (an element of A) is not associated to any element of B. But, according to the definition of a function, all the elements of A should be associated with elements of B. Hence, f_4 is not a function from A to B.

Note

Every function is a relation but its converse is not true.

e.g. Let us consider $R_1 = \{(1, 3), (1, 5), (2, 5)\}$

and $R_2 = \{(2, 5), (3, 4)\}$.

Here, set R_1 and R_2 both are relations but R_1 is not a function because element 1 has two images and set R_2 is a function because no two elements has same image.

EXAMPLE |3| If $x, y \in \{1, 2, 3, 4\}$, then check whether f_1, f_2 and f_3 are functions or not where

$$f_1 = \{(x, y) : y = x + 1\},$$
$$f_2 = \{(x, y) : x + y = 5\}$$
and $$f_3 = \{(x, y) : x + y > 4\}.$$

Also find the the range in case of a function.

Sol. Let $A = \{1, 2, 3, 4\}$, then $f_1 f_2$ and f_3 are defined from A to A.

First express f_1, f_2 and f_3 as sets of ordered pairs, i.e. write $f_1 = \{(1, 2), (2, 3), (3, 4)\}$

$f_2 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

and $f_3 = \{(1, 4), (2, 3), (2, 4), (3,3), (3,2), (3,4)$
$(4, 1), (4, 2), (4, 3), (4, 4)\}$

Now in f_1, we observe that an element $4 \in A$ is not appeared at first place of any ordered pair of f_1. So, f_1 is not a function from A to A. In f_2, we observe that each element of set A is appeared at first place in one and only one ordered pair of f_2. So, f_2 is a function from A to A and Range of $f_2 = \{1, 2, 3, 4\}$. In f_3, we observe that $2, 3, 4 \in A$ have appeared at first place of ordered pair more than one time. So, f_3 is not a function.

Domain, Codomain and Range of a Function

If f is a function from A to B, i.e. each element of A corresponds to one and only one element of B, whereas every element in B need not be the image of some x in A. Then the set A is called the **domain** of function f and the set B is called the **codomain** of f.

The subset of B containing the images of elements of A is called the **range** of the function.

Thus, if a function f is expressed as the set of ordered pairs, then the domain of f is the **set of first elements** of all ordered pairs of f and the range of f is **the set of second elements** of all ordered pairs of f i.e.

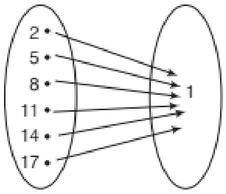
$$\text{Domain of } f = \{a : (a, b) \in f\}$$
$$\text{Range of } f = \{b : (a, b) \in f\}$$

EXAMPLE |4| Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range. **[NCERT]**

(i) $\{(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)\}$
(ii) $\{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)\}$
(iii) $\{(1, 3), (1, 5), (2, 5)\}$

Sol. Here,

(i) $\{(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)\}$

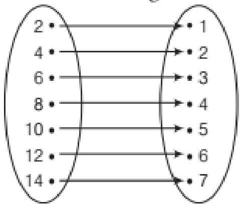


It is clear from arrow diagram that

(*a*) it is a function, as each element have a unique image.
(*b*) its domain is $\{2, 5, 8, 11, 14, 17\}$.
(*c*) its range is $\{1\}$.

(ii) $\{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)\}$.

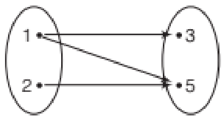
It is clear from arrow diagram that



(a) it is a function, as each element have a unique image.
(b) its domain is $\{2, 4, 6, 8, 10, 12, 14\}$.
(*c*) its range is $\{1, 2, 3, 4, 5, 6, 7\}$.

(iii) $\{(1, 3), (1, 5), (2, 5)\}$

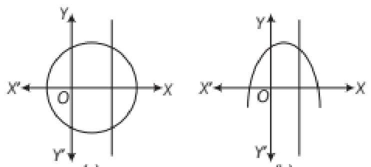
It is clear from the arrow diagram that it is not a function because 1 has two images 3 and 5.



Identification of a Function with its Graph

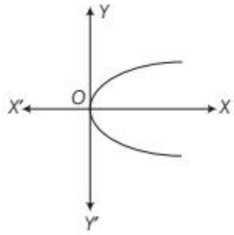
We draw a vertical line i.e. *any line parallel to Y*-axis. If this line intersect the graph of the curve in more than one point, then the curve is a relation and if it intersect at only one point, then the **curve is a function**.

In figure (a), the vertical line intersects the curve at two points, therefore the curve is a relation whereas in figure (b), the vertical line intersects the curve at only one point, therefore the curve is a function.

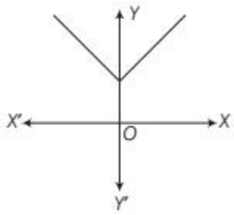


(a) (b)

EXAMPLE |5| Identify the curve, which is a relation or function.

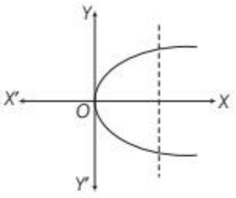


(i)



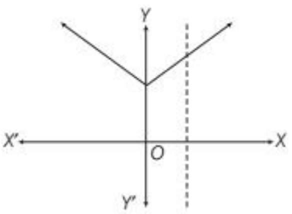
(ii)

Sol. (*i*) If we draw a vertical line, then it will intersect the curve at two points. It shows that given curve is a relation.



(i)

(*ii*) If we draw a vertical line, then it will intersect the curve at only one point. It shows that given curve is a function.



(ii)

Equal Functions

Two functions f and g are said to be equal iff

(i) domain of f = domain of g

(ii) codomain of f = codomain of g

(iii) $f(x) = g(x)$ for every x belonging to their common domain and then we write $f = g$.

e.g. Let $A = \{1, 2\}, B = \{3, 6\}$ and $f : A \to B$ is given by $f(x) = x^2 + 2$ and $g : A \to B$ given by $g(x) = 3x$. Here, we observe that f and g have the same domain and codomain.

Also, $\quad f(1) = 3 = g(1)$

and $\quad f(2) = 6 = g(2)$

Hence, $\quad f = g$

EXAMPLE |6| Let $A = \left\{-2, \frac{1}{2}\right\}$ and $B = \left\{7, \frac{-1}{2}\right\}$, $f : A \to B$ defined by $f(x) = 2x^2 - 1$ and $g : A \to B$ defined by $g(x) = 1 - 3x$. Find whether $f = g$ or not.

Sol. We observe that f and g both have same domain and codomain. Also, we have

$$f(-2) = 7 = g(-2) \text{ and } f\left(\frac{1}{2}\right) = \frac{-1}{2} = g\left(\frac{1}{2}\right)$$

Hence, $f = g$

EXAMPLE |7| $f : R - \{3\} \to R$ be defined by $f(x) = \dfrac{x^2 - 9}{x - 3}$ and $g : R \to R$ be defined by $g(x) = x + 3$. Find whether $f = g$ or not.

Sol. Here, $f(x) = \dfrac{x^2 - 9}{x - 3} = \dfrac{(x - 3)(x + 3)}{x - 3} = (x + 3)$

and $\quad g(x) = x + 3 \Rightarrow f(x) = g(x) = x + 3$

But $f(x) \neq g(x)$ as domain of $f(x)$ is $R - \{3\}$ and domain of $g(x)$ is R.

$\Rightarrow$ Domain of $f(x) \neq$ Domain of $g(x) \Rightarrow f \neq g$

EXAMPLE |8| Find the values of x for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal.

Sol. Given, $\quad f(x) = 3x^2 - 1$ and $g(x) = 3 + x$

Let $\quad f(x) = g(x)$

Then, $\quad 3x^2 - 1 = 3 + x \Rightarrow 3x^2 - x - 4 = 0$

$\Rightarrow 3x^2 - 4x + 3x - 4 = 0 \Rightarrow x(3x - 4) + 1(3x - 4) = 0$

$\Rightarrow \quad (3x - 4)(x + 1) = 0 \Rightarrow x = -1, \frac{4}{3}$

Real Functions

A function which has either R (set of real numbers) or one of its subsets as its range is called a **real valued function** and if its domain is also either R or a subset of R i.e. range and domain both are either R or a subset of R then function is called **real function.**

For real functions, generally domain and codomain both are infinite subsets of R. Therefore, a real function is generally described by some general formula. In other words we can say that, images of various elements of the domain of a real function are provided by some general formula. *e.g.* Let f be a function define by $f(x) = 2x + 1$ from N to N. Then,

x	1	2	3	4	5
f (x) = y	f (1) = 3	f (2) = 5	f (3) = 7	f (4) = 9	f (5) = 11

Thus, range $= \{3, 5, 7, 9, 11, \ldots\} \subseteq R$, codomain $= N$ and domain $= N \subseteq R$.

EXAMPLE |9| If $f(x) = x^2 + 2x + 3$, then find $f(1)$, $f(2)$ and $f(3)$.

Substitute $x = 1, 2$ and 3 in $f(x)$ and get the corresponding values of $f(1)$, $f(2)$ and $f(3)$, respectively.

Sol. We have, $f(x) = x^2 + 2x + 3$ ...(i)

$$f(1) = (1)^2 + 2(1) + 3 \quad \text{[putting } x = 1 \text{ in Eq. (i)]}$$
$$= 1 + 2 + 3 = 6$$
$$f(2) = (2)^2 + 2(2) + 3 \quad \text{[putting } x = 2 \text{ in Eq. (i)]}$$
$$= 4 + 4 + 3 = 11$$
$$f(3) = (3)^2 + 2(3) + 3 \quad \text{[putting } x = 3 \text{ in Eq. (i)]}$$
$$= 9 + 6 + 3 = 18$$

EXAMPLE |10| If $f(1 + x) = x^2 + 1$, then find $f(2 - h)$.

Substitute $x = 1 - h$ in given function and get the value of $f(2 - h)$.

Sol. We have, $f(1 + x) = x^2 + 1$ ...(i)

On substituting $x = (1 - h)$ in Eq. (i), we get

$$f(1 + 1 - h) = (1 - h)^2 + 1$$
$$f(2 - h) = 1 + h^2 - 2h + 1$$
$$= h^2 - 2h + 2$$

EXAMPLE |11| If $y = f(x) = \frac{1 - x}{1 + x}$, then show that $x = f(y)$.

Sol. We have, $y = f(x) = \frac{1 - x}{1 + x}$ ...(i)

Then, $$f(y) = \frac{1 - y}{1 + y} = \frac{1 - \left(\frac{1 - x}{1 + x}\right)}{1 + \left(\frac{1 - x}{1 + x}\right)} \quad \text{[from Eq. (i)]}$$

$$= \frac{\frac{1 + x - (1 - x)}{1 + x}}{\frac{1 + x + 1 - x}{1 + x}} = \frac{2x}{2}$$

$\Rightarrow \quad f(y) = x$

$\therefore \quad x = f(y)$, when $y = f(x) = \frac{1 - x}{1 + x}$

EXAMPLE |12| If f is a real function defined by $f(x) = \frac{x - 1}{x + 1}$, then prove that $f(2x) = \frac{3f(x) + 1}{f(x) + 3}$

Sol. Given, $f(x) = \frac{x - 1}{x + 1}$ ...(i)

$\Rightarrow \quad f(2x) = \frac{2x - 1}{2x + 1}$ ...(ii)

Now, consider $$\frac{3f(x) + 1}{f(x) + 3} = \frac{3\left(\frac{x - 1}{x + 1}\right) + 1}{\frac{x - 1}{x + 1} + 3} \quad \text{[from Eq. (i)]}$$

$$= \frac{\frac{3(x - 1) + (x + 1)}{(x + 1)}}{\frac{(x - 1) + 3(x + 1)}{(x + 1)}} = \frac{3x - 3 + x + 1}{x - 1 + 3x + 3} = \frac{4x - 2}{4x + 2}$$

$$= \frac{2(2x - 1)}{2(2x + 1)} = \frac{2x - 1}{2x + 1} \quad \text{...(iii)}$$

From Eq. (ii) and (iii), we get

$$f(2x) = \frac{3f(x) + 1}{f(x) + 3}$$ **Hence proved.**

EXAMPLE |13| If the function t which maps temperature in degree Celcius into temperature in degree Fahrenheit is defined by $t(C) = \frac{9C}{5} + 32$, then find **[NCERT]**

(i) $t(0)$ (ii) $t(28)$ (iii) $t(-10)$

(iv) the value of C, when $t(C) = 212$

Sol. Given, $t(C) = \frac{9C}{5} + 32$ ...(i)

(i) On putting $C = 0$ in Eq. (i), we get

$$t(0) = \frac{9 \times 0}{5} + 32 = 0 + 32 = 32$$

(ii) On putting $C = 28$ in Eq. (i), we get

$$t(28) = \frac{9 \times 28}{5} + 32 = \frac{252}{5} + \frac{32}{1}$$
$$= \frac{252 + 160}{5} = \frac{412}{5}$$

(iii) On putting $C = -10$ in Eq. (i), we get

$$t(-10) = \frac{9 \times (-10)}{5} + 32$$
$$= \frac{-9 \times 10}{5} + 32 = -9 \times 2 + 32$$
$$= -18 + 32 = 14$$

(iv) Now, on putting $t(C) = 212$ in Eq. (i), we get

$$212 = \frac{9C}{5} + 32$$

$\Rightarrow \quad \frac{9C}{5} = 212 - 32$

$\Rightarrow \quad \frac{9C}{5} = 180$

$\because \quad C = \frac{5 \times 180}{9}$

$\therefore \quad C = 5 \times 20 = 100$

EXAMPLE |14| Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to Z$ given by $f(x) = x^2 - 2x - 3$

Find (i) the range of f (ii) pre-image of 6, −3, 5.

Sol. (i) $f(-2) = (-2)^2 - 2\times(-2) - 3 = 5$

$f(-1) = (-1)^2 - 2(-1) - 3 = 0$

$f(0) = -3$

$f(1) = 1^2 - 2\times 1 - 3 = -4$

$f(2) = 2^2 - 2\times 2 - 3 = -3$

So, range $(f) = \{-4, -3, 0, 5\}$

(ii) Let x be the pre-image of 6.

$\Rightarrow \quad f(x) = 6$

$\Rightarrow \quad x^2 - 2x - 3 = 6$

$\Rightarrow \quad x^2 - 2x - 9 = 0$

$$\Rightarrow \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\times 1\times(-9)}}{2\times 1} = \frac{2\pm\sqrt{4+36}}{2} = \frac{2\pm\sqrt{40}}{2}$$

$\Rightarrow \quad x = 1 \pm \sqrt{10} \notin A$

$\therefore$ 6 does not have pre-image in A.

Let x be the pre-image of −3.

$\therefore \quad f(x) = -3$

$\Rightarrow \quad x^2 - 2x - 3 = -3$

$\Rightarrow \quad x^2 - 2x = 0$

$\Rightarrow \quad x(x - 2) = 0$

$\Rightarrow \quad x = 0$ or $2 \in A$

So, 0 and 2 are pre-image of −3.

Again let x be the pre-image of 5.

$\therefore \quad f(x) = 5$

$\Rightarrow \quad x^2 - 2x - 3 = 5$

$\Rightarrow \quad x^2 - 2x - 8 = 0$

$\Rightarrow \quad x^2 - 4x + 2x - 8 = 0$

$\Rightarrow \quad x(x-4) + 2(x-4) = 0$

$\Rightarrow \quad (x-4)(x+2) = 0$

$\Rightarrow \quad x = 4$ or -2

Since, $-2 \in A$. So, −2 is the pre-image of 5.

EXAMPLE |15| Given $A = \{-1, 0, 2, 5, 6, 11\}$, $B = \{-2, -1, 0, 18, 28, 108\}$ and $f(x) = x^2 - x - 2$. Is $f(A) = B$? Find $f(A)$.

Sol. Given, $f(x) = x^2 - x - 2$,

$A = \{-1, 0, 2, 5, 6, 11\}$

and $B = \{-2, -1, 0, 18, 28, 108\}$

Clearly, $f(-1) = (-1)^2 - (-1) - 2 = 0$

$f(0) = (0)^2 - 0 - 2 = -2$

$f(2) = (2)^2 - 2 - 2 = 0$

$f(5) = (5)^2 - 5 - 2 = 18$

$f(6) = (6)^2 - 6 - 2 = 28$

and $f(11) = (11)^2 - 11 - 2 = 108$

Now, $f(A) = \{f(x) : x \in A\}$

$= \{0, -2, 18, 28, 108\}$

Here, we see that $-1 \in B$ but $-1 \notin f(A)$

$\therefore \quad f(A) \neq B$.

EXAMPLE |16| $f = (1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a function describe by the formula $f(x) = ax + b$ for some integers a, b. Determine a, b.

Sol. Given, $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$

Here, $f(1) = 1$, $f(2) = 3$, $f(0) = -1$, $f(-1) = -3$

It is given that $f(x) = ax + b$

$\therefore \quad f(1) = 1$ and $f(2) = 3$

$\Rightarrow \quad a + b = 1$ and $2a + b = 3$

On solving both equations, we get

$a = 2, b = -1$

Thus, $f(x) = 2x - 1$

This also satisfy $f(0) = -1$

and $f(-1) = -3$

Hence, $a = 2, b = -1$

Domain of Real Functions

As we know that real functions are generally described by providing the general formula for finding the images of elements of its domain.

In such cases, the domain of the real function $f(x)$ is the set of all those real numbers for which the expression for $f(x)$ or the formula for $f(x)$ assumes real values only.

In other words, we can say that the domain of $f(x)$ is the set of all those real numbers for which $f(x)$ is meaningful.

WORKING RULE FOR FINDING THE DOMAIN OF REAL FUNCTIONS

The domain of a real function depends on the nature of function. e.g. Polynomial function is always define for all real numbers, so its domain is R. Depending upon function, we may use the following steps to determine their domain.

For Rational Functions

Step I First put the denominator equal to zero and find the values of variable.

Step II The value of variable obtained in step I are the values at which rational function is not defined. Then, omit the set of these values from R to get required domain.

EXAMPLE |17| Find the domain of real function

$$f(x)=\frac{x^2+3x+5}{x^2-5x+4}$$

Sol. Here, denominator $= x^2-5x+4$

On putting it equal to zero, we get

$$x^2-5x+4=0$$

$\Rightarrow \quad x^2-4x-x+4=0$

$\Rightarrow \quad x(x-4)-1(x-4)=0$

$\Rightarrow \quad (x-4)(x-1)=0$

$\Rightarrow \quad x=4 \text{ or } x=1$

Clearly, at $x=1$ and $x=4$ denominator becomes zero and thus given function will not be define for these value.

So, required domain of given function $= R-\{1, 4\}$

For square root function

We know that expression under the square root should not be negative. So, for finding the domain of square root function, put the expression without root greater than equal to zero and then find the value of variable for which it is positive.

EXAMPLE |18| Find the domain of each of the following function

(i) $\frac{1}{\sqrt{x-2}}$ (ii) $\sqrt{4-x^2}$

Sol. (i) Let $f(x)=\frac{1}{\sqrt{x-2}}$

$f(x)$ assumes real values if $x-2>0$

[here, we take only greater than sign, because it is in rational form and so denominator should not be equal to zero]

$\Rightarrow \quad x>2 \Rightarrow x\in(2,\infty).$

Hence, domain of f is $(2,\infty)$.

(ii) Given, $f(x)=\sqrt{4-x^2}$

Clearly, $f(x)$ assumes real values, if $4-x^2\ge 0$

$\Rightarrow \quad -(x^2-4)\ge 0 \Rightarrow x^2-4\le 0$

[multiplying by -1 on both sides]

$\Rightarrow \quad (x-2)(x+2)\le 0$

$[\because a^2-b^2=(a-b)(a+b)]$

$\Rightarrow \quad x\in[-2, 2]$

Hence, domain of $f=[-2, 2]$

EXAMPLE |19| Find the domain for which the functions $f(x)=2x^2-1$ and $g(x)=1-3x$ are equal.

[NCERT Exemplar]

Sol. Given, $\quad f(x)=2x^2-1$

and $\quad g(x)=1-3x$

Since, $\quad f(x)=g(x)$

$\therefore \quad 2x^2-1=1-3x$

$\Rightarrow \quad 2x^2+3x-2=0$

$\Rightarrow \quad 2x^2+4x-x-2=0$

$\Rightarrow \quad 2x(x+2)-1(x+2)=0$

$\Rightarrow \quad (2x-1)(x+2)=0$

$\Rightarrow \quad 2x-1=0 \text{ or } x+2=0$

$\Rightarrow \quad x=\frac{1}{2} \text{ or } x=-2$

Thus, domain for which the function

$$f(x)=g(x) \text{ is } \left\{\frac{1}{2}, -2\right\}$$

EXAMPLE |20| Find the domain of $f(x)=\frac{1}{\sqrt{x+|x|}}$.

Sol. We have, $\quad f(x)=\frac{1}{\sqrt{x+|x|}}$

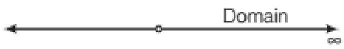
We know that, $|x|=\begin{cases} x, & \text{if } x\ge 0 \\ -x, & \text{if } x<0 \end{cases}$

$\therefore \quad x+|x|=\begin{cases} x+x, & \text{if } x\ge 0 \\ x-x, & \text{if } x<0 \end{cases}$

$\Rightarrow \quad x+|x|=\begin{cases} 2x, & \text{if } x\ge 0 \\ 0, & \text{if } x<0 \end{cases}$ ...(i)

Here, the function $f(x)=\frac{1}{\sqrt{x+|x|}}$ assumes real values, if $x+|x|>0$.

$\Rightarrow \quad 2x>0 \Rightarrow x>0$ [using Eq. (i)]



$\therefore \quad x\in(0,\infty)$

Hence, domain of $f=(0,\infty)$.

Note

Domain of modulus function is always R i.e. $(-\infty, \infty)$.

Range of Real Functions

The range of a real function of a real variable is the set of all real values taken by $f(x)$ at points of its domain.

WORKING RULE FOR FINDING RANGE OF REAL FUNCTIONS

Let $y=f(x)$ be a real function, then for finding the range we may use the following steps

***Step* I** Find the domain of the function $y=f(x)$.

***Step* II** Transform the equation $y=f(x)$ as $x=g(y)$.
i.e. convert x in terms of y.

***Step* III** Find the values of y from $x=g(y)$ such that the values of x are real and lying in the domain of f.

***Step* IV** The set of values of y obtained in step III be the range of function f.

EXAMPLE |21| Find the range of $f(x) = \frac{x-2}{3-x}$.

Sol. Given, $f(x) = \frac{x-2}{3-x} = y$ (say)

Here, denominator $= 3 - x$

Put denominator $= 0$, we get

$$3 - x = 0 \Rightarrow x = 3$$

$\therefore$ Domain of $y = f(x)$ is $R - \{3\}$.

Now, consider $y = \frac{x-2}{3-x}$

$\Rightarrow$ $y(3-x) = (x-2)$

$\Rightarrow$ $3y - xy = x - 2$

$\Rightarrow$ $x + xy = 3y + 2$

$\Rightarrow$ $x(1+y) = 3y + 2$

$\Rightarrow$ $x = \frac{3y+2}{1+y}$

It will define, if $1 + y \neq 0 \Rightarrow y \neq -1$.

So, $x = g(y)$ takes real values, if $y \in R - \{-1\}$.

Also, $\frac{3y+2}{1+y} \neq 3$ for all $y \in R - \{-1\}$

[$\because$ if $\frac{3y+2}{1+y} = 3$, then $3y + 2 = 3 + 3y$

$\Rightarrow 2 = 3$, which is absurd]

$\Rightarrow$ $x = \frac{3y+2}{1+y} \neq 3$ for all $y \in R - \{-1\}$

Thus, $x = \frac{3y+2}{1+y} \in$ Domain (f), $\forall\ y \in R - \{1\}$

Hence, the range of $f(x)$ is $R - \{-1\}$.

EXAMPLE |22| Find the domian and range of $f(x) = \left\{\left(x, \frac{1}{1-x^2}\right) : x \in R\ x \neq \pm 1\right\}$.

Sol. Given, $f(x) = \left\{\left(x, \frac{1}{1-x^2}\right) : x \in R,\ x \neq \pm 1\right\}$

Clearly, $f(x) = \frac{1}{1-x^2}$ is not defined, when $(1 - x^2) = 0$

i.e. when $x = \pm 1$.

$\therefore$ Domain of $f(x) = R - \{-1, 1\}$

Further, let $y = \frac{1}{1-x^2}$

$\Rightarrow$ $(1 - x^2) = \frac{1}{y} \Rightarrow x^2 = 1 - \frac{1}{y}$

$\Rightarrow$ $x = \pm\sqrt{\left(1 - \frac{1}{y}\right)} = \pm\sqrt{\frac{y-1}{y}}$

Clearly, x is define, if $\frac{y-1}{y} \geq 0$ and $y \neq 0$

$(y-1)\, y \geq 0$ and $y \neq 0$

+ – +
0 1

$\Rightarrow$ $-\infty < y < 0$ or $1 \leq y < \infty$

Also, $\sqrt{\frac{y-1}{y}} \neq 1$ for all $y \in (-\infty, 0) \cup [1, \infty)$

$\Rightarrow$ $x = \pm\sqrt{\frac{y-1}{y}} \neq \pm 1$ for all $y \in (-\infty, 0) \cup [1, \infty)$

[$\because$ if $\sqrt{\frac{y-1}{y}} = 1$, $y - 1 = y \Rightarrow -1 = 0$, which is observed]

Thus, $x = \pm\sqrt{\frac{y-1}{y}} \in$ domain (f), $\forall\ y \in (-\infty, 0) \cup [1, \infty)$

Hence, range $(f) = (-\infty, 0) \cup [1, \infty)$

EXAMPLE |23| Find the domain and range of the function

$$f(x) = \left\{\left(x, \frac{x^2-1}{x-1}\right) : x \in R, x \neq 1\right\}$$

Sol. Let $f(x) = \left\{\left(x, \frac{x^2-1}{x-1}\right) : x \in R,\ x \neq 1\right\}$

Clearly, f is not defined, when $x = 1$

So, f is defined for all real values of x, except $x = 1$.

$\therefore$ Domain $(f) = R - \{1\}$

Let $y = \frac{x^2-1}{x-1}$

$= \frac{(x-1)(x+1)}{(x-1)} = x + 1$ [as $x \neq 1$]

$\Rightarrow$ $x = y - 1$

Clearly, x is not defined, when $y = 2$ as $x \neq 1$.

Range $(f) = R - \{2\}$

EXAMPLE |24| Let $f : R \to R$ be such that $f(x) = 2^x$. Determine

(i) range of f

(ii) whether $f(x + y) = f(x) \cdot f(y)$ holds.

(iii) $\{x : f(x) = 1\}$

Sol. (i) Given, $f(x) = 2^x$, which is well-defined for all $x \in R$.

$\therefore$ Domain $(f) = R$

Now, let $y = f(x) = 2^x$

Taking log with 2 on both sides, we get

$$\log_2 y = \log_2 2^x$$

$\Rightarrow$ $\log_2 y = x \log_2 2 = x$ [$\because \log_a a = 1$]

$\Rightarrow x = \log_2 y$, which is define for all $y > 0$.

Hence, the range of f is set of all positive real numbers.

(ii) We have, $f(x+y) = 2^{x+y} = 2^x \cdot 2^y = f(x) \cdot f(y)$

$\therefore$ $f(x+y) = f(x) \cdot f(y)$ holds for all $x, y \in R$.

(iii) $\because$ $f(x) = 1$

$2^x = 1 \Rightarrow x = 0 \quad \therefore \{x : f(x) = 1\} = \{0\}$

EXAMPLE |25| Find the domain and the range of the function $f(x) = \sqrt{16 - x^2}$.

Domain is the collection of those values of x for which $f(x)$ is defined and range $= \{f(x) : \forall\, x \in \text{Domain}\}$.

Sol. Given, $f(x) = \sqrt{16 - x^2}$

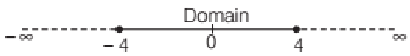
Domain of f We observe that $f(x)$ is defined for all x satisfying $16 - x^2 \geq 0$

$\Rightarrow \quad x^2 - 16 \leq 0$

$\Rightarrow \quad (x-4)(x+4) \leq 0 \quad [\because a^2 - b^2 = (a-b)(a+b)]$

$\Rightarrow \quad -4 \leq x \leq 4$

$\Rightarrow \quad x \in [-4, 4]$



Range of f Let $y = f(x)$. Then,

$y = \sqrt{16 - x^2} \Rightarrow y^2 = 16 - x^2$

$\Rightarrow \quad x^2 = 16 - y^2$

$\Rightarrow \quad x = \pm\sqrt{16 - y^2}$

Clearly, x will take real values, if

$16 - y^2 \geq 0 \Rightarrow y^2 - 16 \leq 0$

$\Rightarrow (y-4)(y+4) \leq 0 \Rightarrow -4 \leq y \leq 4$

$\Rightarrow \quad y \in [-4, 4]$

Also, $\quad y = \sqrt{16 - x^2} \geq 0$ for all $x \in [-4, 4]$

$\therefore \quad y \in [0, 4]$ for $x \in [-4, 4]$

Hence, range of f [0, 4].

EXAMPLE |26| Find the domain and range of the function $f(x) = \dfrac{1}{\sqrt{x-5}}$.

Sol. We have, $f(x) = \dfrac{1}{\sqrt{x-5}}$

Domain of f

Clearly, $f(x)$ takes real values, if

$x - 5 > 0 \Rightarrow x > 5 \Rightarrow x \in (5, \infty)$

Domain of $f = (5, \infty)$

Range of f

For $x > 5$, we have $x - 5 > 0$

$\Rightarrow \quad \sqrt{x-5} > 0$

$\Rightarrow \quad \dfrac{1}{\sqrt{x-5}} > 0 \Rightarrow f(x) > 0$

Thus, $f(x)$ takes all real values greater than zero.

Hence, range of $f = (0, \infty)$.

EXAMPLE |27| Find the domain and range of the real valued function $f(x)$ given by $f(x) = \dfrac{4-x}{x-4}$

Sol. Given, $f(x) = \dfrac{4-x}{x-4}$

Domain of f Here, we see that $f(x)$ is defined for all x except $= 4$

$\therefore \quad \text{Domain}(f) = R - \{4\}$

$\because \quad f(x) = \dfrac{4-x}{x-4} = \dfrac{-(x-4)}{x-4} = -1$

$\therefore \quad \text{Range}(f) = \{-1\}$

EXAMPLE |28| Find the domain and range of the function $f(x) = 1 - |x-2|$.

Sol. Given function is $f(x) = 1 - |x-2|$.

We know that, $|x|$ is defined for $x \in (-\infty, \infty)$.

$\therefore$ Domain of $|x-2|$ is $(-\infty, \infty)$.

Hence, domain of $f(x) = 1 - |x-2|$ is $(-\infty, \infty)$.

Also we know that, range of $|x|$ is $[0, \infty)$.

$\therefore$ Range of $|x-2|$ is $[0, \infty)$.

$\therefore \quad 0 \leq |x-2| < \infty$

$\Rightarrow \quad -\infty < -|x-2| \leq 0$

$\Rightarrow -\infty + 1 < 1 - |x-2| \leq 1 \Rightarrow -\infty < f(x) \leq 1$

$\therefore$ Range of $f(x)$ is $(-\infty, 1]$.

EXAMPLE |29| Find the domain and range of the function $f(x) = \dfrac{1}{2 - \sin 3x}$.

Sol $\because \quad -1 \leq \sin 3x \leq 1 \;\forall\; x \in R$

$\Rightarrow \quad -1 \leq -\sin 3x \leq 1, \forall x \in R$

$\Rightarrow \quad 1 \leq 2 - \sin 3x \leq 3, \forall x \in R$

$\Rightarrow \quad 2 - \sin 3x \neq 0$ for any $x \in R$

$\Rightarrow \quad f(x) = \dfrac{1}{2 - \sin 3x}$ is defined, $\forall\; x \in R$.

$\therefore \quad \text{Domain}(f) = R$

From the above discussion, $1 \leq 2 - \sin 3x \leq 3, \forall\; x \in R$

$\Rightarrow \quad \dfrac{1}{3} \leq \dfrac{1}{2 - \sin 3x} \leq 1, \forall x \in R$

$\Rightarrow \dfrac{1}{3} \leq f(x) \leq 1, \forall\; x \in R \Rightarrow f(x) \in \left[\dfrac{1}{3}, 1\right]$

$\therefore \quad \text{Range}(f) = \left[\dfrac{1}{3}, 1\right]$

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 The domain for which the functions defined by $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal to

(a) $\left\{-1, \frac{4}{3}\right\}$ (b) $\left\{-1, -\frac{4}{3}\right\}$

(c) $\left\{1, \frac{4}{3}\right\}$ (d) $\left\{1, -\frac{4}{3}\right\}$

2 Let $f(x) = \sqrt{1+x^2}$, then **[NCERT Exemplar]**

(a) $f(xy) = f(x) \cdot f(y)$

(b) $f(xy) \geq f(x) \cdot f(y)$

(c) $f(xy) \leq f(x) \cdot f(y)$

(d) None of the above

3 Domain of $\sqrt{a^2 - x^2}$ $(a > 0)$ is **[NCERT Exemplar]**

(a) $(-a, a)$ (b) $[-a, a]$

(c) $[0, a]$ (d) $(-a, 0]$

4 If $f(x) = \frac{1}{2 - \sin 3x}$, then range (f) is equal to

(a) $[-1, 1]$ (b) $\left[-\frac{1}{3}, \frac{1}{3}\right]$

(c) $\left[\frac{1}{3}, 1\right]$ (d) $\left[-1, \frac{-1}{3}\right]$

VERY SHORT ANSWER Type Questions

5 Which of the relations are functions?

(i) $f_1 = \{(2,1), (3,1), (4,1), (3,2), (4,2), (4,3)\}$

(ii) $f_2 = \{(1,4), (2,3), (3,2), (4,1)\}$

6 Let f be a relation on the set N of natural numbers defined by $f = \{(n, 3n) : n \in N\}$. Is f a function from N to N. If so, find the range of f.

7 Let $f : Z \to Z$, $g : Z \to Z$ be functions defined by

$f = \{(x, x^2) : x \in Z\}$

$g = \{(x, |x|^2) : x \in Z\}$, show that $f = g$

8 Let $f : R \to R$ be given by $f(x) = x^2 + 3$.
Find $\{x : f(x) = 28\}$

9 In question 4, find the pre-images of 39 and 2 under f.

10 If $f(x) = 3x^4 - 5x^2 + 9$, find $f(x-1)$.

11 Let f be defined by $f(x) = (x - 4)$ and g be defined by $g(x) = \begin{cases} \frac{x^2 - 16}{x+4}, & x \neq -4 \\ \lambda, & x = -4 \end{cases}$

Find λ such that $f(x) = g(x)$ for all x.

SHORT ANSWER Type I Questions

12 If $f: R \to R$ be defined as follows

$$f(x) = \begin{cases} 1, & x \in Q \\ -1, & x \notin Q \end{cases}$$

Find (i) $f\left(\frac{1}{2}\right)$, $f(\pi)$ (ii) pre-image of 1 and –1.

13 A function $f : R \to R$ is defined by $f(x) = x^2$.
Determine

(i) Range of f (ii) $\{x : f(x) = 4\}$

14 If $A = \{1, 2, 3, 4\}$, $B = \{1, 2, \ldots, 10\}$ and $f : A \to B$ is a function defined by $f(x) = 2x + 1$, $x \in A$, then find the range of f.

15 Find the domain of the function

$$f(x) = \frac{x}{x^2 + 3x + 2}.$$

16 If $f(x) = x + \frac{1}{x}$, prove that $[f(x)]^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$

17 Is $g = \{(1,1), (2,3), (3,5), (4,7),\}$ a function, justify? If this is described by the relation, $g(x) = \alpha x + \beta$, then what values should be assigned to α and β? **[NCERT Exemplar]**

SHORT ANSWER Type II Questions

18 If $f(x) = \frac{1}{2x+1}$, $x \neq -\frac{1}{2}$, then show that

$$f(f(x)) = \frac{2x+1}{2x+3}, \text{ provided that } x \neq -\frac{3}{2}.$$

19 Find the range of the following function.

$$f(x) = \frac{x}{1+x^2}$$

20 Let f and g be real functions defined by $f(x) = 2x + 1$ and $g(x) = 4x - 7$.

(i) For what real numbers x, $f(x) = g(x)$?

(ii) For what real numbers x, $f(x) < g(x)$?

[NCERT Exemplar]

21 Let $f = \left\{\left(x, \frac{x^2}{1+x^2}\right) : x \in R\right\}$ be a functions R into R determine the range of R. **[NCERT]**

HINTS & ANSWERS

1. (a) Given, $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$

$\because \quad 3x^2 - 1 = 3 + x$

$\Rightarrow \quad 3x^2 - x - 4 = 0$

$\Rightarrow \quad 3x^2 - 4x + 3x - 4 = 0$

$\Rightarrow \quad x(3x - 4) + 1(3x - 4) = 0$

$\Rightarrow \quad (x + 1)(3x - 4) = 0$

$\Rightarrow \quad x = -1, \; x = \frac{4}{3}$

2. (c) $f(x) = \sqrt{1 + x^2}$

Now, $f(xy) = \sqrt{1 + x^2y^2}$ ...(i)

and $f(x)f(y) = \sqrt{1 + x^2}\sqrt{1 + y^2}$

$\Rightarrow \quad f(x)f(y) = \sqrt{1 + x^2y^2 + x^2 + y^2}$ ...(ii)

$\therefore$ From Eqs. (i) and (ii), we get

$f(x)f(y) \geq f(xy)$

3. (b) Let $f(x) = \sqrt{a^2 - x^2}$

For $f(x)$ to be defined $a^2 - x^2 \geq 0 \Rightarrow x^2 \leq a \Rightarrow x \in [-a, a]$

4. (c) We know that, $-1 \leq \sin 3x \leq 1$

$\Rightarrow \quad -1 \leq -\sin 3x \leq 1$

$\Rightarrow \quad 1 \leq 2 - \sin 3x \leq 3$

$\Rightarrow \quad \frac{1}{3} \leq \frac{1}{2 - \sin 3x} \leq 1$

$\Rightarrow \quad \frac{1}{3} \leq f(x) \leq 1$

$\therefore$ Range $(f) = \left[\frac{1}{3}, 1\right]$

5. (i) It is not a function because the first elements 3 and 4 of ordered pairs have more than one images.

(ii) It is a function bacause first element of each ordered pair has one and only one image.

6. Since, for each $n \in N$. There exist a unique $3n \in N$ such $(n, 3n) \in f$. Therefore f is a function from N to N

Now, range $(f) = \{3n : n \in N\}$

7. Here, domain (f) = domain (g)

codomain f = codomain g

Now, $f(x) = x^2$ and $g(x) = |x|^2 = x^2 \quad (\because |x|^2 = x^2)$

$\Rightarrow \quad f(x) = g(x), \; \forall \; x \in Z$

8. Given, $f(x) = x^2 + 3$ and $f(x) = 28$

$\therefore \quad x^2 + 3 = 28 \Rightarrow x^2 = 28 - 3$

$\Rightarrow \quad x^2 = 25 \Rightarrow x = \pm 5$

$\Rightarrow \quad x \in \{-5, 5\}$

9. Let x be the pre-image of 39

$\Rightarrow \quad f(x) = 39 \Rightarrow x^2 + 3 = 39$

$\Rightarrow \quad x^2 = 36 \Rightarrow x = \pm 6$

Again, let pre-image of 2 be x.

$\Rightarrow \quad f(x) = 2 \Rightarrow x^2 + 3 = 2$

$\Rightarrow \quad x^2 = -1$ [which is not possible]

$\therefore$ 2 does not have any pre-image.

10. $f(x) = 3x^4 - 5x^2 + 9$

$\therefore \quad f(x - 1) = 3(x - 1)^4 - 5(x - 1)^2 + 9$

$= 3x^4 - 12x^3 + 13x^2 - 2x + 7$

11. Given, $f(x) = g(x), \forall \; x \in R$

$\Rightarrow \quad f(-4) = g(-4)$

$\Rightarrow \quad -4 - 4 = \lambda \Rightarrow \lambda = -8$

12. (i) The value of the function for every rational number is 1 and for every irrational number is -1.

$\therefore \quad \frac{1}{2} \in Q \Rightarrow f\left(\frac{1}{2}\right) = 1$

and $\quad \pi \notin Q \Rightarrow f(\pi) = -1$

(ii) $\because \quad f(x) = 1 \; \forall \; x \in Q$

$\therefore$ pre-image of 1 is rational numbers

$\because \quad f(x) = -1 \; \forall \; x \notin Q$

$\therefore$ Pre-image of -1 is all real number except rational number i.e. $R - Q$ = Set of irrational numbers

13. (i) $f(x) = x^2$, clearly $\forall \; x \in R$, $f(x) = x^2 \geq 0$

$\therefore$ Range $= [0, \infty)$

(ii) $f(x) = 4 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \Rightarrow x \in \{-2, 2\}$

14. Given, $f(x) = 2x + 1$

$\therefore \quad f(1) = 2 \times 1 + 1 = 2 + 1 = 3 \in B$

$\Rightarrow \quad f(2) = 2 \times 2 + 1 = 4 + 1 = 5 \in B$

$\Rightarrow \quad f(3) = 2 \times 3 + 1 = 6 + 1 = 7 \in B$

$\Rightarrow \quad f(4) = 2 \times 4 + 1 = 8 + 1 = 9 \in B$

$\therefore \quad$ Range $f = \{3, 5, 7, 9\}$

15. $f(x) = \frac{x}{x^2 + 3x + 2} = \frac{x}{(x + 1)(x + 2)}$

Clearly, $f(x)$ is not defined at $x = -1, -2$.

$\therefore \quad$ Domain $(f) = R - \{-1, -2\}$.

16. $\because \quad f(x) = x + \frac{1}{x} \Rightarrow f(x^3) = x^3 + \frac{1}{x^3}$

Now, $[f(x)]^3 = \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$

$[\because (a + b)^3 = a^3 + b^3 + 3ab(a + b)]$

$= f(x^3) + 3f(x)$

$= f(x^3) + 3f\left(\frac{1}{x}\right) \quad \left[\because f(x) = f\left(\frac{1}{x}\right)\right]$

17. We have, $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$

Since, every element has unique image under g. So, g is a function.

Now, $\quad g(x) = \alpha x + \beta$

When $x = 1$, then $g(1) = \alpha(1) + \beta$ ...(i)

$\Rightarrow$ $1 = \alpha + \beta$

When $x = 2$, then $g(2) = \alpha(2) + \beta$

$\Rightarrow$ $3 = 2\alpha + \beta$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$\alpha = 2, \beta = -1$$

18. $f(f(x)) = f\left(\frac{1}{2x+1}\right) = \frac{1}{2\left(\frac{1}{2x+1}\right)+1} = \frac{2x+1}{2x+3}$

Clearly, it is not defined at $x \neq -\frac{3}{2}$.

19. Let $y = f(x) = \frac{x}{1+x^2}$; $yx^2 - x + y = 0$

$\Rightarrow$ $x = \frac{1 \pm \sqrt{1-4y^2}}{2y}$

Clearly, x will assume real values, if

$$1 - 4y^2 \geq 0 \text{ and } y \neq 0$$

$\Rightarrow$ $y^2 - \frac{1}{4} \leq 0$ and $y \neq 0$

$\Rightarrow$ $-\frac{1}{2} \leq y \leq \frac{1}{2}$ and $y \neq 0$

$\Rightarrow$ $y \in \left[\frac{-1}{2}, \frac{1}{2}\right] - \{0\}$

Also, $y = 0$ for $x = 0$

Hence, range $(f) = \left[-\frac{1}{2}, \frac{1}{2}\right]$

20. We have, $f(x) = 2x + 1$ and $g(x) = 4x - 7$

(i) $\because$ $f(x) = g(x)$

$\therefore$ $2x + 1 = 4x - 7$

$\Rightarrow$ $2x = 8$

$\Rightarrow$ $x = 4$

(ii) $\because$ $f(x) < g(x)$

$\therefore$ $2x + 1 < 4x - 7$

$\Rightarrow$ $2x + 1 < -7$

$\Rightarrow$ $-2x < -8$

$\Rightarrow$ $x > 4$

21. $f(x)$ is defined for all $x \in R$. As $x^2 + 1 \neq 0$ for any $x \in R$.

Now, let $y = f(x) = \frac{x^2}{x^2+1}$

$\Rightarrow$ $x^2 = \frac{y}{1-y} \Rightarrow x = \pm\sqrt{\frac{y}{1-y}}$

Clearly, x will be real, if

$$\frac{y}{1-y} \geq 0 \text{ and } y \neq 1$$

$\Rightarrow$ $\frac{y}{y-1} \leq 0$ and $y \neq 1 \Rightarrow 0 \leq y < 1$

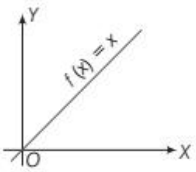
$\therefore$ Range $(f) = [0, 1)$.

|TOPIC 4|
Some Standard Real Functions and their Graphs

Here, we shall study some important real functions, which are frequently used in the study of calculus.

Identity Function

The real function $f : R \to R$ defined by $f(x) = x, \forall x \in R$ is called an **identity function**. Its domain is R and range is also R.



It may be observed that

(i) Graph of identity function is a straight line.

(ii) it passes through origin.

Constant Function

A function $f : R \to R$ is said to be a **constant function**, if there exists a **real number** k, such that

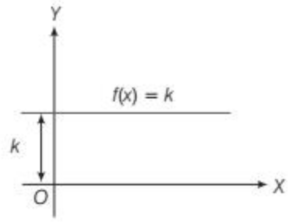
$$f(x) = k \text{ for all } x \in R$$

Here, domain of $f = R$ and its range $= \{k\}$

e.g., $f(x) = 5, \forall x \in R$ (symbol '$\forall$' means for all)

Here, $f(x)$ is a constant function whose

domain $= \{x : x \in R\}$ and range $= \{5\}$



It may be observed that

(*i*) The graph of a constant function is a line parallel to X-axis.

(*ii*) If $k > 0$, then the line will be above X-axis.

(*iii*) If $k = 0$, then graph coincides with the X-axis.

(*iv*) If $k < 0$, then the line will be below X-axis.

EXAMPLE |1| Draw the graph of each the following function

(i) $f(x) = 5$ (ii) $f(x) = -5$

(iii) $f(x) = 0$

Sol. (i) Given, $f(x) = 5 \ \forall \ x \in R$

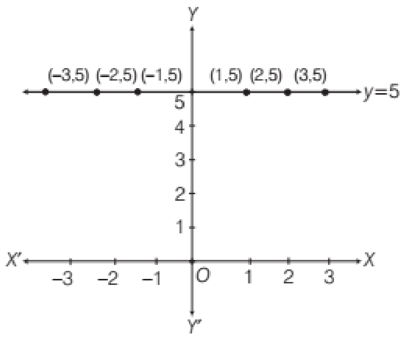
Clearly, domain of f is R.

Range of f is $\{5\}$.

Now, we prepare a table for different value of x as shown below

x	−3	−2	−1	0	1	2	3
***f*(x)**	5	5	5	5	5	5	5

Locate the points (−3, 5), (−2, 5), (−1, 5), (0, 5), (1, 5), (2, 5) and (3, 5) on the graph paper and joint them, we obtained the required graph as shown below.



Clearly the graph of $f(x) = 5$ represents a straight line parallel to X-axis and at a distance of 5 unit above the X-axis and symmetric with respect to Y-axis.

(ii) Given, $f(x) = -5$

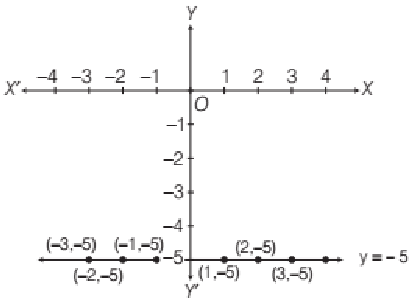
Clearly, domain of $f(x)$ is R.

Range of $f(x)$ is $\{-5\}$.

Now, we prepare a table for different values of x as shown below.

x	−3	−2	−1	0	1	2	3
***f*(x)**	−5	−5	−5	−5	−5	−5	−5

Locate the points $(-3, -5)$, $(-2, -5)$, $(-1, -5)$, $(0, -5)$, $(1, -5)$, $(2, -5)$, $(3, -5)$ on the graph paper and join them we obtained the graph of the function as shown below.



Clearly, the graph of $f(x) = -5$ represent a straight line parallel to X-axis and at a distance of 5 unit below the X-axis and symmetric with respect to Y-axis.

(iii) Given, $f(x) = 0$

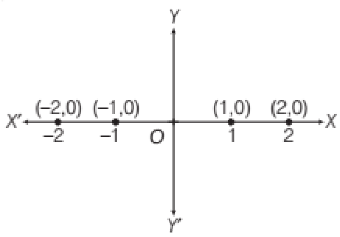
Clearly, domain f is R.

Range f is $\{0\}$,

We have the following table for different value of x.

x	−2	−1	0	1	2
f(*x*)	0	0	0	0	0

Now, locate the point $(-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0)$ on the graph paper, we get the required graph of the function.



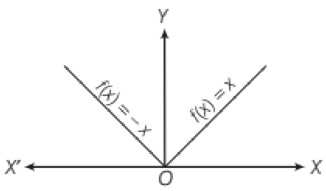
Clearly, the graph of $f(x) = 0$ represent X-axis and symmetric with respect to Y-axis.

Modulus Function or Absolute Value Function

The function $f : R \to R$ defined by

$$f(x) = |x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

is called the modulus function. It is also called **absolute value function**. Its domain is R and its range is $[0, \infty)$.



It may be observed that

(*i*) The graph is symmetrical with respect to Y-axis.
(*ii*) Graph lies above the X-axis.
(*iii*) It passes through the origin.
(*iv*) In the first quadrant, it coincides with the graph of the identity function.

PROPERTIES OF MODULUS FUNCTION

(*i*) For any real number x, we have $\sqrt{x^2} = |x|$

(*ii*) If a and b are positive real numbers, then

(a) $x^2 \le a^2 \Leftrightarrow |x| \le a \Leftrightarrow -a \le x \le a$

(b) $x^2 \ge a^2 \Leftrightarrow |x| \ge a \Leftrightarrow x \le -a$ or $x \ge a$

(c) $x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$

(d) $x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a$ or $x > a$

(e) $a^2 \le x^2 \le b^2 \Leftrightarrow a \le |x| \le b$
$\Leftrightarrow x \in [-b, -a] \cup [a, b]$

(f) $a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b$
$\Leftrightarrow x \in (-b, -a) \cup (a, b)$

EXAMPLE |2| The function f is defined by

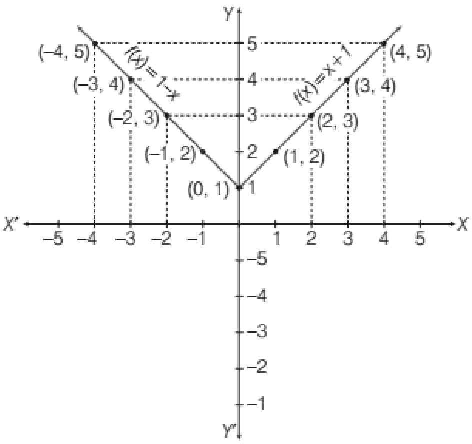
$$f(x) = \begin{cases} 1 - x, & x < 0 \\ 1, & x = 0 \\ x + 1, & x > 0 \end{cases}$$

Draw the graph of $f(x)$. **[NCERT]**

Sol. Here, $f(x) = 1 - x, x < 0$, this gives

$f(-4) = 1 - (-4) = 5$; $f(-3) = 1 - (-3) = 4$
$f(-2) = 1 - (-2) = 3$; $f(-1) = 1 - (-1) = 2$
and $f(x) = x + 1, x > 0$; $f(1) = 2, f(2) = 3, f(3) = 4$;
$f(4) = 5$ and so on.

Now, the graph of f is as shown in following figure.



EXAMPLE |3| Draw the graph of the function $f : R \to R$ such that $f(x) = |x - 2|$.

Define the function for $x < 2$ and $x \ge 2$ and draw the graph.

Sol. Clearly, $y = |x - 2| = \begin{cases} x - 2, & \text{if } x - 2 \ge 0 \\ -(x - 2), & \text{if } x - 2 < 0 \end{cases}$

$$= \begin{cases} x - 2, & \text{if } x \ge 2 \\ 2 - x, & \text{if } x < 2 \end{cases}$$

We know that, a linear equation in x and y represents a line. For drawing a line, we need only two points.

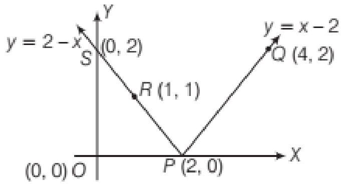
For $y = x - 2$

x	2	4
y	0	2

So, plot the points $P(2, 0)$, $Q(4, 2)$ and join PQ to get the graph of $y = x - 2$. For $y = 2 - x$

x	1	0
y	1	2

Plot the points $R(1, 1)$, $S(0, 2)$ and join RS to get the graph of $y = 2 - x$.



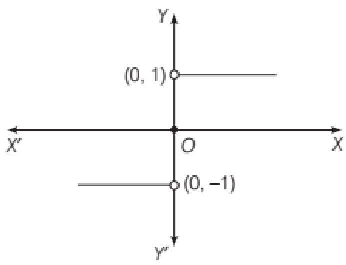
Signum Function

The function $f : R \to R$ defined by

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{when } x \ne 0 \\ 0, & \text{when } x = 0 \end{cases} \quad \text{or} \quad f(x) = \begin{cases} 1, & \text{when } x > 0 \\ 0, & \text{when } x = 0 \\ -1, & \text{when } x < 0 \end{cases}$$

is called the **signum function.**

Therefore, its domain is R, range is $\{-1, 0, 1\}$ and its graph is given below



Greatest Integer Function (Step Function)

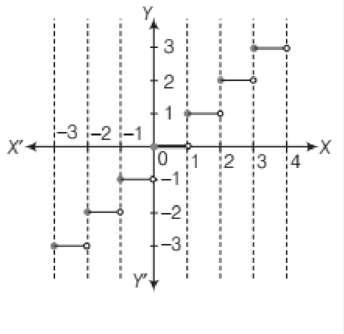
The function $f: R \to R$ defined by $f(x) = [x]$ is called the greatest integer function, where $[x] =$ **integral** part of x or greatest integer less than or equal to x.

Its domain is R and its range is I (I = set of integer).

e.g., $[7.4] = [7 + 0.4] = 7$; $[0.6] = [0 + 0.6] = 0$

$[-1.5] = [-2 + 0.5] = -2$

Graph



Values of x	$f(x) = [x]$
⋮	⋮
$-3 \le x < -2$	-3
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2
$3 \le x < 4$	3
⋮	⋮

It may be observed that

(i) $[x]$ = [Integer + Proper positive fraction] = Integer

(ii) It passes through origin.

(iii) It is symmetrical in the opposite quadrant.

PROPERTIES OF GREATEST INTEGER FUNCTION

If n is an integer and x is a real number between n and $n+1$, then

(i) $[-n] = -n$

(ii) $[x + k] = [x] + k$ for any integer k.

(iii) $[-x] = -[x] - 1$

(iv) $[x] + [-x] = \begin{cases} -1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $[x] - [-x] = \begin{cases} 2[x] + 1, & \text{if } x \notin Z \\ 2[x], & \text{if } x \in Z \end{cases}$

(vi) $[x] \ge k \Rightarrow x \ge k$, where $k \in Z$.

(vii) $[x] \le k \Rightarrow x < k + 1$, where $k \in Z$.

(viii) $[x] > k \Rightarrow x \ge k + 1$, where $k \in Z$.

(ix) $[x] < k \Rightarrow x < k$, where $k \in Z$.

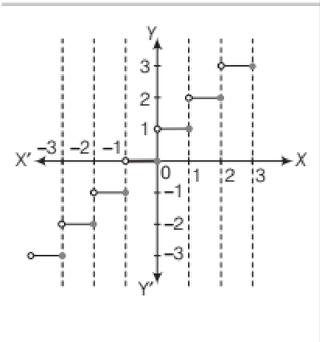
Least (Smallest) Integer Function (Ceiling Function)

The function $f: R \to R$ defined by $f(x) = \lceil x \rceil$ for all $x \in R$ is called the smallest integer function or the ceiling function. It is also a **step function**. Its domain is R and range is I.

For any real number x, we use the symbol $\lceil x \rceil$ or (x) to denote the smallest integer greater than or equal to x.

e.g. $\lceil 5.9 \rceil = 6, \lceil -7.2 \rceil = -7$, etc.

Graph



Values of x	$f(x) = \lceil x \rceil$
⋮	⋮
$-3 < x \le -2$	-2
$-2 < x \le -1$	-1
$-1 < x \le 0$	0
$0 < x \le 1$	1
$1 < x \le 2$	2
$2 < x \le 3$	3
⋮	⋮

It may be observed that

(*i*) It passes through origin

(*ii*) It is symmetrical in the opposite quadrants.

PROPERTIES OF SMALLEST INTEGER FUNCTION

Following are some properties of smallest integer function

(*i*) $\lceil -n \rceil = -\lceil n \rceil$, where $n \in Z$

(*ii*) $\lceil -x \rceil = -\lceil x \rceil + 1$, where $x \in R - Z$

(*iii*) $\lceil x + n \rceil = \lceil x \rceil + n$, where $x \in R - Z$ and $n \in Z$

(*iv*) $\lceil x \rceil + \lceil -x \rceil = \begin{cases} 1, \text{ if } x \notin Z \\ 0, \text{ if } x \in Z \end{cases}$

(*v*) $\lceil x \rceil - \lceil -x \rceil = \begin{cases} 2\lceil x \rceil - 1, & \text{if } x \notin Z \\ 2\lceil x \rceil, & \text{if } x \in Z \end{cases}$

Polynomial Function

A function $f: R \to R$, defined by

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2, \dots, a_n \in R$, is called a polynomial function.

If $a_n \neq 0$, then n is called the **degree of the polynomial.**

The domain of a polynomial function is R and range depends on the polynomial representing the function. e.g.,

(i) $f(x) = x^3 - x^2 + 9$ is polynomial function in x.

(ii) $h(x) = x^5 + \sqrt{3}x$ is polynomial function in x.

(iii) $g(x) = x^{\frac{3}{2}} + 3x$ is not a polynomial function. Since, power $\frac{3}{2}$ is not an integer.

EXAMPLE |4| Let R be the set of real numbers. Define the real function $f: R \to R$ by $f(x) = x + 10$. Sketch the graph of this function.

Sol. Given, $f: R \to R$ define by $f(x) = x + 10$.

Here, at $x = 0$, $f(0) = 10$

at $x = 1$, $f(1) = 11$

at $x = 2$, $f(2) = 12$

... ...

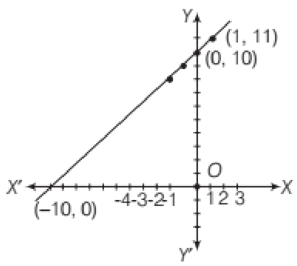
at $x = 10$, $f(10) = 20$ and so on

Also, at $x = -1$, $f(-1) = 9$

at $x = -2$, $f(-2) = 8$

at $x = -10$, $f(-10) = 0$ and so on

On plotting the points (0, 10), (1, 11), (2, 12)... (10, 20), (−1, 9), (−2, 8) and (−10, 0) on graph paper, we get the following graph



This is an example of linear function.

Note

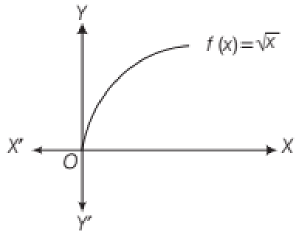
The function f defined by $f(x) = mx + c$, $x \in R$ is called **linear function**, where m and c are constants.

Square Root Function

The square root of a function is defined as $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and range is R^+ i.e. $[0, \infty)$.

We observe that the values of $f(x) = \sqrt{x}$ increase with the increase in x.

So, its graph is given below



It may be observe that it is passing through origin.

Rational Function

A function of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions of x defined in a domain and $Q(x) \neq 0$, is called a rational function.

The domain of a rational function is $R - \{x : Q(x) = 0\}$ and range depends on the expression representing the function.

Thus, $f(x) = \left(\frac{x^2 + 4}{x^3 - 6x + 4}\right)$ is called a rational function, where $x^3 - 6x + 4 \neq 0$.

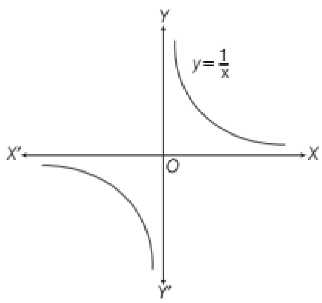
Its domain is $R - \{x : x^3 - 6x + 4 = 0\}$.

Reciprocal Function

The function $f : R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$ is called the reciprocal function. Its domain as well as range is $R - \{0\}$.

We observe that the sign of $\frac{1}{x}$ is same as that of x and $\frac{1}{x}$ decreases with the increase in x.

So, its graph is given below



It is also a rational function.

Exponential Function

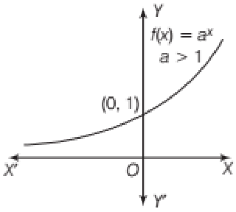
Let $a(\neq 1)$ be a positive real number. Then, the function $f: R \to R$, defined by $f(x) = a^x$, is called the exponential function. Its domain is R and range is $(0, \infty)$ as it attains only positive values. As, $a > 0$ and $a \neq 1$.

So following cases aries

***Case* I** When $a > 1$, then the values of $y = f(x) = a^x$ increase as the values of x increase.

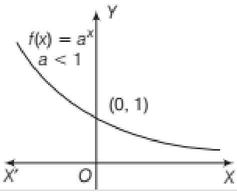
Thus, $f(x) = a^x = \begin{cases} <1, \text{ for } x < 0 \\ 1, \text{ for } x = 0 \\ >1, \text{ for } x > 0 \end{cases}$

Its graph is given below

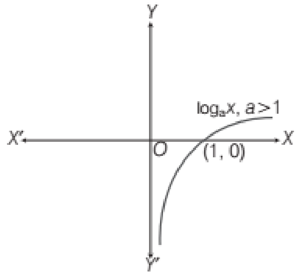


***Case* II** When $0 < a < 1$, then the values of $y = f(x) = a^x$ decrease with the increase in x and $y > 0$ for all $x \in R$. Thus, $f(x) = a^x = \begin{cases} >1, \text{ for } x < 0 \\ 1, \text{ for } x = 0 \\ <1, \text{ for } x > 0 \end{cases}$

Its graph is given below



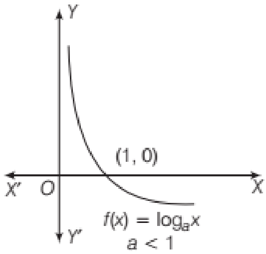
Its graph is given below



***Case* II** When $0 < a < 1$, then values of y decreases with the increase in x.

So, $y = \log_a x = \begin{cases} >0, & \text{for } 0 < x < 1 \\ 0, & \text{for } x = 1 \\ <0, & \text{for } x > 1 \end{cases}$

Its graph is given below



PROPERTIES OF LOGARITHMIC FUNCTION

(*i*) $\log_a 1 = 0$, where $a > 0$, $a \neq 1$

(*ii*) $\log_a a = 1$, where $a > 0$, $a \neq 1$

(*iii*) $\log_a (xy) = \log_a (x) + \log_a (y)$, where $a > 0$, $a \neq 1$ and $x, y > 0$.

Logarithmic Function

Let $a(\neq 1)$ be a positive real number. Then, the function $f: (0, \infty) \to R$, defined by $f(x) = \log_a x$, is called the logarithmic function. Its domain is $(0, \infty)$ and range is R.

As, $a > 0$ and $a \neq 1$. *So following cases aries*

***Case* I** When $a > 1$, then values of y increases with the increase in x. So, $y = \log_a x = \begin{cases} <0, \text{ for } x < 1 \\ 0, \text{ for } x = 1 \\ >0, \text{ for } x > 1 \end{cases}$

where $a > 0$, $a \neq 1$, and $x, y > 0$.

(*v*) $\log_a x^m = m \log_a x$, where $a > 0$, $a \neq 1$ and $x > 0$

(*vi*) $\log_{a^n} x^m = \frac{m}{n} \log_a (x)$, where $a > 0$, $a \neq 1$ and $x > 0$

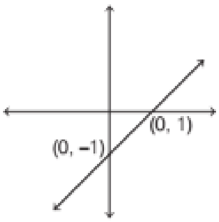
Note

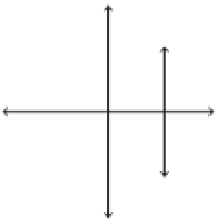
Functions $\log_a x$ and a^x are inverse of each other. So, their graphs are mirror images of each other in the line $y = x$.

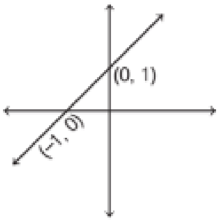
TOPIC PRACTICE 4

OBJECTIVE TYPE QUESTIONS

1 The graph of the function, $f(x) = x - 1$ is

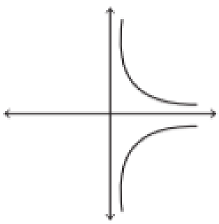
(a) 

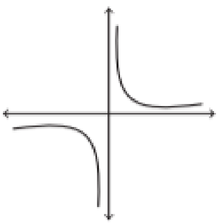
(b) 

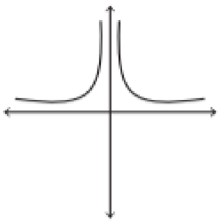
(c) 

(d) None of these

2 The graph of the function, $f(x) = \frac{1}{x^2}$ is

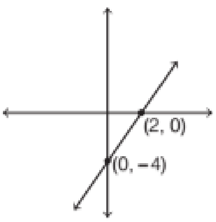
(a) 

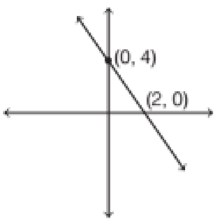
(b) 

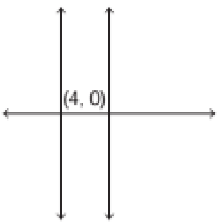
(c) 

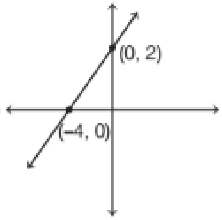
(d) None of these

3 The graph of the function, $f(x) = 4 - 2x$ is

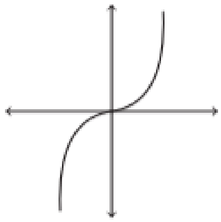
(a) 

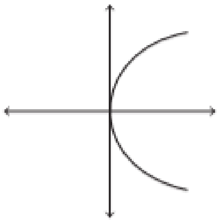
(b) 

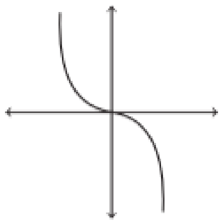
(c) 

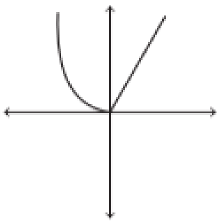
(d) 

4 The graph of the function, $f(x) = x^3$ is

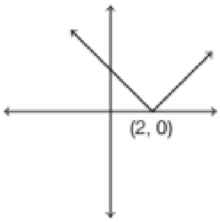
(a) 

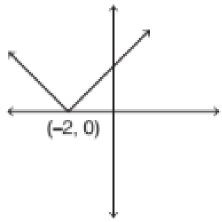
(b) 

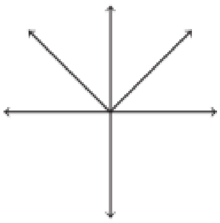
(c) 

(d) 

5 The graph of the functions, $f(x) = |x - 2|$ is

(a) 

(b) 

(c) 

(d) None of these

SHORT ANSWER Type I Questions

Draw the graph of the following function. Also, determine their domain and range.

6 $f(x) = 2$

7 $f(x) = -2$

8 $f(x) = |x - 3|$

9 $f(x) = \begin{cases} x^2, & x \geq 0 \\ x, & x < 0 \end{cases}$

SHORT ANSWER Type II Questions

10 $f(x) = 1 - x^2$

11 Let $f: R \to R$ defined by $f(x) = 1 - x^2$ for all $x \in R^+$. Find its domain and range. Also, draw its graph.

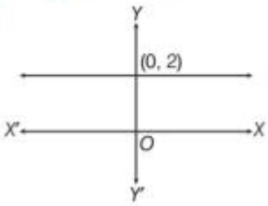
12 Draw the graph of $f(x) = \frac{1}{2}x^3$. Also, find it domain and range.

HINTS & ANSWERS

1. (a) **2.** (c) Clearly, the domain of the function is $R - \{0\}$.

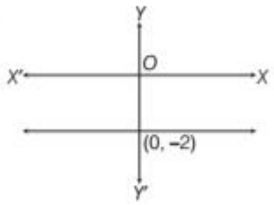
3. (b) **4.** (a) **5.** (a)

6.



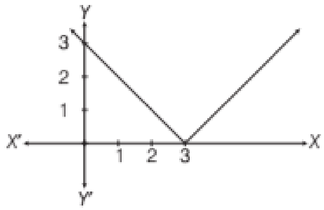
Domain $(f) = R$; Range $(f) = \{2\}$

7.



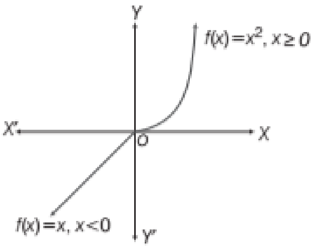
Domain $(f) = R$; Range $(f) = \{-2\}$

8.



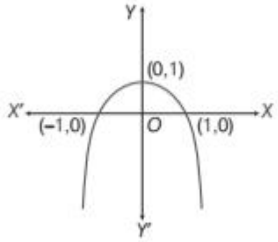
Domain $(f) = R$; Range $(f) = [0, \infty)$

9.



Domain $(f) = R$; Range $(f) = R$

10.



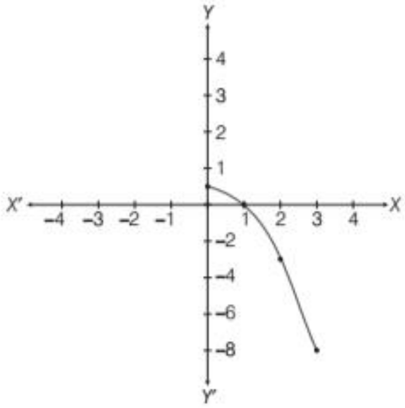
Domain $(f) = R$; Range $(f) = (-\infty, 1]$

11. $f(x) = 1 - x^2$

$\therefore \forall x \in R^+$

Domain of $f = (0, \infty)$ and Range of $f = (-\infty, 1)$

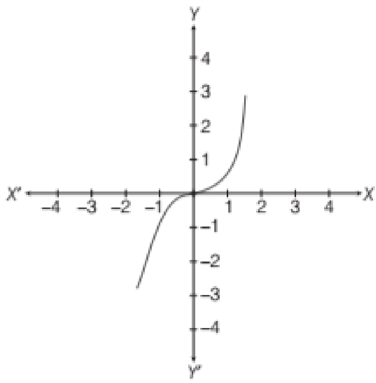
x	0	1	2	3
f(x)	1	0	–3	–8



12. $f(x) = \frac{1}{2}x^3 \forall x \in R$

x	–2	0	2
f(x)	–4	0	4

Doamin of $f = (-\infty, \infty)$; Range of $f = (-\infty, \infty)$



|TOPIC 5|
Algebra of Real Functions

Let $f: D_1 \to R$ and $g: D_2 \to R$ be two real functions with domain D_1 and D_2, respectively. Then, algebraic operations such as addition, subtraction, multiplication, division and scalar multiplication on two real functions are given below

(i) **Addition of two real functions** The sum function $(f+g)$ is defined by

$$(f+g)(x) = f(x) + g(x), \forall x \in D_1 \cap D_2$$

The domain of $(f+g)$ is $D_1 \cap D_2$.

(ii) **Subtraction of two real functions** The difference function $(f-g)$ is defined by

$$(f-g)(x) = f(x) - g(x), \forall x \in D_1 \cap D_2$$

The domain of $(f-g)$ is $D_1 \cap D_2$.

(iii) **Multiplication of two real functions** The product function (fg) is defined by

$$(fg)(x) = f(x) \cdot g(x), \forall x \in D_1 \cap D_2$$

The domain of (fg) is $D_1 \cap D_2$.

(iv) **Quotient of two real functions** The quotient function is defined by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \forall x \in D_1 \cap D_2 - \{x : g(x) = 0\}$$

The domain of $\left(\frac{f}{g}\right)$ is $D_1 \cap D_2 - \{x : g(x) = 0\}$.

(v) **Multiplication of a real function by a scalar** The scalar multiple function cf is defined by

$$(cf)(x) = c \cdot f(x), \forall x \in D_1$$

where, c is a scalar (real number).

The domain of cf is D_1.

Note For any real function $f: D \to R$ and $n \in N$, we define

$$(\underbrace{fff \ldots f}_{n \text{ times}})(x) = \underbrace{f(x) f(x) \ldots f(x)}_{n \text{ times}} = \{f(x)\}^n, \forall x \in D$$

EXAMPLE |1| If two functions are defined as $f(x) = \frac{1}{(x-2)}$, $x \neq 2$ and $g(x) = (x-2)^2$, then find

(i) $f+g$ (ii) $f-g$ (iii) fg (iv) $\frac{f}{g}$

Sol. Given functions are

$$f(x) = \frac{1}{x-2}, x \neq 2 \text{ and } g(x) = (x-2)^2$$

(i) $(f+g)(x) = f(x) + g(x) = \frac{1}{x-2} + (x-2)^2, x \neq 2$

$$= \frac{1+(x-2)^3}{x-2}, x \neq 2$$

(ii) $(f-g)(x) = f(x) - g(x) = \frac{1}{(x-2)} - (x-2)^2, x \neq 2$

$$= \frac{1-(x-2)^3}{(x-2)}, x \neq 2$$

(iii) $(fg)(x) = f(x) \cdot g(x) = \frac{1}{(x-2)} \times (x-2)^2, x \neq 2$

$$= (x-2), x \neq 2$$

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\left(\frac{1}{x-2}\right)}{(x-2)^2}, x \neq 2 = \frac{1}{(x-2)^3}, x \neq 2$

EXAMPLE |2| Find the sum and difference of the identity function and the modulus function.

If f and g are two functions, then their sum is defined by $(f+g)(x) = f(x) + g(x)$, $\forall x \in D_1 \cap D_2$, where D_1 and D_2 are domains of f and g, respectively.

Sol. We know that,

$f: R \to R$ defined by $f(x) = x$ is the identity function.

$g: R \to R$ defined by $g(x) = |x|$ is the modulus function.

$\therefore$ Sum, $(f+g): R \to R$ defined by

$$(f+g)(x) = f(x) + g(x) = x + |x|$$

$$= \begin{cases} x+x, & \text{if } x \geq 0 \\ x-x, & \text{if } x < 0 \end{cases} = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Difference, $(f-g): R \to R$ defined by

$$(f-g)(x) = f(x) - g(x) = x - |x|$$

$$= \begin{cases} x-x, & \text{if } x \geq 0 \\ x-(-x), & \text{if } x < 0 \end{cases} = \begin{cases} 0, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$$

EXAMPLE |3| Find the quotient of the identity function by the modulus function.

Sol. Let f and g denote respectively the identity function and the modulus function. Then,

$f: R \to R$ is defined as $f(x) = x$

$f: R \to R$ is defined as $f(x) = x$

and $g: R \to R$ is defined as $g(x) = |x|$.

Clearly, f and g have the same domain.

Also, $g(x) = 0 \Rightarrow |x| = 0 \Rightarrow x = 0$.

Therefore, the quotient of f by g i.e. $\frac{f}{g}$ is a function from $R - \{0\} \to R$ and it is defined as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{|x|} = \begin{cases} \frac{x}{x} = 1, & x > 0 \\ \frac{x}{-x} = -1, & x < 0 \end{cases}$$

EXAMPLE |4| Find the product of the identity function and the reciprocal function.

Sol. Let f and g denote respectively the identity function and the reciprocal function. Then,

$f: R \rightarrow R$ is defined as $f(x) = x$ for all $x \in R$ and

$g: R - \{0\} \rightarrow R$ and is defined as $g(x) = \frac{1}{x}$ for all $x \in R - \{0\}$

We have, domain $(f) \cap$ domain (g)

$$= R \cap R - \{0\} = R - \{0\}$$

Therefore, the product fg is a function from $R - \{0\}$ to R and is defined as

$(fg)(x) = f(x)g(x) = x \times \frac{1}{x} = 1$ for all $x \in R - \{0\}$

EXAMPLE |5| Let λ be a non-zero number and $f: R \rightarrow R$ be a function defined by $f(x) = \frac{x}{\lambda}$ $\forall$ $x \in R$ Find.

(i) λf (ii) $\lambda^2 f$ (iii) $\left(\frac{1}{\lambda}\right) f$

Sol. Clearly, $\lambda f, \lambda^2 f$ and $\left(\frac{1}{\lambda}\right) f$ are functions from R to itself

(i) $(\lambda f)(x) = \lambda \cdot f(x) = \lambda \times \frac{x}{\lambda} = x, \forall\ x \in R$

(ii) $(\lambda^2 f)(x) = \lambda^2 \cdot f(x) = \lambda^2 \times \frac{x}{\lambda} = \lambda x, \forall\ x \in R$

(iii) $\left(\frac{1}{\lambda} f\right)(x) = \frac{1}{\lambda} \cdot f(x) = \frac{1}{\lambda} \times \frac{x}{\lambda} = \frac{x}{\lambda^2}, \forall\ x \in R$

EXAMPLE |6| Let f be the exponential function and g be the logarithmic function. Find

(i) $(f + g)(1)$ (ii) $(f \cdot g)(1)$
(iii) $(3f)(1)$ (iv) $(5g)(1)$

Sol. We have, $f: R \rightarrow R$ given by $f(x) = e^x$ and $g: R \rightarrow R$ given by $g(x) = \log_e x$

(i) Since, domain $(f) \cap$ domain $(g) = R \cap R^+ = R^+$

$\therefore f + g: R^+ \rightarrow R$ is defined as

$(f + g)(x) = f(x) + g(x) = e^x + \log_e x \forall\ x \in R^+$

Clearly, $1 \in R^+$

$\therefore\ (f + g)(1) = f(1) + g(1) = e^1 + \log_e 1 = e + 0 = e$ $[\because \log 1 = 0]$

(ii) $f \cdot g: R^+ \rightarrow R$ and is defined as

$(f.g)(x) = f(x)\, g(x) = e^x \log_e x$

$(f.g)(1) = f(1)\, g(1) = e^1 \log_e 1 = 0$

(iii) $(3f)(x) = 3 \cdot (f(x)) = 3e^x$; $(3f)(1) = 3 \cdot e^1 = 3e$

(iv) $(5g)(x) = 5(g(x)) = 5\log_e x$; $(5g)(1) = 5\log_e 1 = 0]$

TOPIC PRACTICE 5

OBJECTIVE TYPE QUESTIONS

1 If $f(x) = \frac{4^x}{4^x + 2}$, then $f\left(\frac{1}{97}\right) + f\left(\frac{2}{97}\right) + \ldots + f\left(\frac{96}{97}\right)$ is equal to

(a) 1 (b) 48
(c) -48 (d) -1

2 If $f(x) = x^3 - \frac{1}{x^3}$, then $f(x) + f\left(\frac{1}{x}\right)$ is equal to

(a) $2x^3$ (b) $\frac{2}{x^3}$
(c) 0 (d) 1

3 If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x) = \frac{(3x + x^3)}{1 + 3x^2}$, then what is $f[g(x)]$ equal to?

(a) $-f(x)$ (b) $3[f(x)]$
(c) $[f(x)]^3$ (d) $-3[f(x)]$

4 If $f(xy) = f(x)\, f(y)$, then $f(t)$ may be of the form

(a) $t + k$ (b) $ct + k$
(c) $t^k + c$ (d) t^k

5 If $x \neq 1$ and $f(x) = \frac{x+1}{x-1}$ is a real function, then $f(f(f(2)))$ equals

(a) 1 (b) 2
(c) 3 (d) 4

SHORT ANSWER Type I Questions

6 Let f and g be real functions defined by $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4 - x^2}$, then find each of the following functions

(i) $f + g$ (i) $f - g$
(iii) $f \cdot g$ (iv) $\frac{f}{g}$

7 Let $f: [2, \infty) \rightarrow R$ and $g: [-2, \infty) \rightarrow R$ be two real functions defined by $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{x+2}$. Find $f + g$ and $f - g$

8 What are the sum and difference of identity function and the reciprocal function? [NCERT]

9 Let $f(x) = x^2$ and $g(x) = 2x + 1$ be two real functions. Find **[NCERT]**

(i) $f + g$ (ii) $f - g$ (iii) $f \cdot g$ (iv) $\frac{f}{g}$

10 Let $f : R \to R$ and $g : R \to R$ defined by $f(x) = x + 1$, $g(x) = 2x - 3$. **[NCERT]**

Find $f + g$, $f - g$, and $\frac{f}{g}$.

SHORT ANSWER Type II Questions

11 If f and g be two real function defined by $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{9 - x^2}$.

Then, describe each of the following functions.

(i) $f + g$ (ii) $g - f$

(iii) $f \cdot g$ (iv) $\frac{f}{g}$

(v) $\frac{g}{f}$ (vi) $2f - \sqrt{5}g$

(vii) $f^2 + 7f$ (viii) $\frac{5}{g}$

12 If $f(x) = \log(1 - x)$ and $g(x) = [x]$, then determine each of the following functions.

(i) $f + g$ (ii) $f.g$ (iii) $\frac{f}{g}$ (iv) $\frac{g}{f}$

Also, find :$(f + g)(-1)$, $(f \cdot g)(0)$, $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$, $\left(\frac{g}{f}\right)\left(\frac{1}{2}\right)$

HINTS & ANSWERS

1. (b) We have, $f(x) = \frac{4^x}{4^x + 2}$

$$\therefore f(1-x) + f(x) = \frac{4^{1-x}}{4^{1-x} + 2} + \frac{4^x}{4^x + 2}$$

$$= \frac{4}{4 + 2 \cdot 4^x} + \frac{4^x}{4^x + 2}$$

[Ist part dividing by 4^{-x}]

$$= \frac{2}{2 + 4^x} + \frac{4^x}{4^x + 2} = \frac{2 + 4^x}{2 + 4^x} = 1$$

Now, putting $x = \frac{1}{97}, \frac{2}{97}, \frac{3}{97}, \ldots, \frac{48}{97}$ and adding, we get

$$\left(f\left(\frac{1}{97}\right) + f\left(\frac{96}{97}\right)\right) + \left(f\left(\frac{2}{97}\right) + f\left(\frac{95}{97}\right)\right) + \ldots + \left(f\left(\frac{48}{97}\right) + f\left(\frac{49}{97}\right)\right)$$

$$= 1 + 1 + \ldots + 1 \quad \text{(48 times)}$$

$$\Rightarrow f\left(\frac{1}{97}\right) + f\left(\frac{2}{97}\right) + \ldots + f\left(\frac{96}{97}\right) = 48$$

2. (c) We have

$$f(x) = x^3 - \frac{1}{x^3}$$

$$f\left(\frac{1}{x}\right) = \frac{1}{x^3} - x^3$$

$$\therefore \quad f(x) + f\left(\frac{1}{x}\right) = x^3 - \frac{1}{x^3} + \frac{1}{x^3} - x^3 = 0$$

3. (b) Given, $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x) = \frac{(3x + x^3)}{1 + 3x^2}$

Now, $f[g(x)] = \log\left(\frac{1 + g(x)}{1 - g(x)}\right)$

$$= \log\left(\frac{1 + 3x^2 + 3x + x^3}{1 + 3x^2 - 3x - x^3}\right)$$

$$= \log\left(\frac{1+x}{1-x}\right)^3$$

$$= 3\log\left(\frac{1+x}{1-x}\right)$$

$$= 3[f(x)]$$

4. (d) Given that,

$$f(xy) = f(x)\, f(y)$$

From option (d), we take $f(t) = t^k$

Then, $f(xy) = (xy)^k = (x^k)(y)^k$

$$= f(x) \cdot f(y)$$

5. (c) We have,

$$f(x) = \frac{x+1}{x-1}$$

$$\therefore \quad f(f(x)) = \frac{f(x) + 1}{f(x) - 1}$$

$$= \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1}$$

$$= \frac{x + 1 + x - 1}{x + 1 - x + 1}$$

$$= \frac{2x}{2} = x$$

$$\therefore \quad f(f(f(x))) = f(x)$$

$$= \frac{x+1}{x-1}$$

$$\Rightarrow \quad f(f(f(2))) = \frac{2+1}{2-1}$$

$$= \frac{3}{1} = 3$$

6. $f(x) = \sqrt{x+2}$, $g(x) = \sqrt{4-x^2}$

$f(x)$ is defined for $x + 2 \geq 0 \Rightarrow x \geq -2$

$\therefore$ Domain $(f) = [-2, \infty)$

$g(x)$ is defined for $4 - x^2 \geq= 0$

$\Rightarrow x^2 - 4 \leq 0$

$\Rightarrow (x-2)(x+2) \leq 0 \Rightarrow x \in [-2, 2]$; Domain $(g) = [-2, 2]$

Domain $(f) \cap$ domain $(g) = [-2, 2]$

Ans. (i) $(\sqrt{x+2})[1 + \sqrt{2-x}]$ (ii) $\sqrt{x+2}\ [1 - \sqrt{2-x}]$

(iii) $(x+2)\sqrt{2-x}$ (iv) $1/\sqrt{2-x}$

7. Domain $(f) = [2, \infty) = D_1$ [say]

Domain $g = [-2, \infty) = D_2$ [say]; $D_1 \cap D_2 = [2, \infty)$

Ans. $(\sqrt{x-2} + \sqrt{x+2})$ and $(\sqrt{x-2} - \sqrt{x+2})$

8. Let $f(x) = x$, $g(x) = \frac{1}{x}$; Domain $(f) = R = D_1$

Domain $(g) = R - \{0\} = D_2$; $D_1 \cap D_2 = R - \{0\}$

Ans. Sum $= x + \frac{1}{x}$ and difference $= x - \frac{1}{x}$

9. $f + g = (x+1)^2$, $f - g = x^2 - 2x - 1$,

$f.g = 2x^3 + x^2$, $\frac{f}{g} = \frac{x^2}{2x+1}$

10. $f + g = 3x - 2$; $f - g = -x + 4$

$\frac{f}{g} : R - \left\{\frac{3}{2}\right\} \to R$, $\frac{f}{g} = \frac{x+1}{2x-3}$

11. Domain $(f) \cap$ Domain $(g) = [-1, 3]$

(i) $(f+g)(x) = \sqrt{x+1} + \sqrt{9-x^2}$

(ii) $(g-f)(x) = \sqrt{9-x^2} - \sqrt{x+1}$

(iii) $(fg)(x) = \sqrt{9 + 9x - x^2 - x^3}$ (iv) $\left(\frac{f}{g}\right)(x) = \sqrt{\frac{x+1}{9-x^2}}$

(v) $\left(\frac{g}{f}\right)(x) = \sqrt{\frac{9-x^2}{x+1}}$, $x \neq -1$

(vi) $(2f - \sqrt{5}g)(x) = 2\sqrt{x+1} - \sqrt{45 - 5x^2}$

(vii) $(f^2 + 7f)(x) = x + 1 + 7\sqrt{x+1}$

(viii) $\left(\frac{5}{g}\right)(x) = \frac{5}{\sqrt{9-x^2}}$, $x \neq -3, 3$

12. (i) $f + g : (-\infty, 1) \to R$ defined by

$(f+g)(x) = \log(1-x) + [x]$

(ii) $fg : (-\infty, 1) \to R$, $(f.g)(x) = [x].\log(1-x)$

(iii) $\frac{f}{g} : (-\infty, 0) \to R$, $\frac{f}{g}(x) = \frac{\log(1-x)}{[x]}$

(iv) $\frac{g}{f} : (-\infty, 0) \cup (0, 1) \to R$, $\left(\frac{g}{f}\right)(x) = \frac{[x]}{\log(1-x)}$

$(f+g)(-1) = \log_e 2 - 1$

$(f \cdot g)(0) = 0$, $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$ = does not exist, $\left(\frac{g}{f}\right)\left(\frac{1}{2}\right) = 0$

SUMMARY

- Let A, B and C be three non-empty sets, then cartesian product of two sets $A \times B = \{(a, b): a \in A, b \in B\}$.
  Here, (a, b) is called an **ordered pair** and cartesian product of three sets $A \times B \times C = \{(a, b, c): a \in A, b \in B, c \in C\}$.
  Here, (a,b,c) is called an **ordered triplet**.
- If $(a, b) = (x, y)$, then $a = x$ and $b = y$.
- If $n(A) = p$ and $n(B) = q$, then
  (i) $n(A \times B) = pq$ (ii) the number of subsets of $A \times B$ is 2^{pq}.
- Let A and B be two non-empty sets. Then, a **relation** R from set A to set B is a subset of cartesian product $A \times B$.
- Let R be the relation from A to B such that $R = \{(a, b): a \in A, b \in B$ and the rule which relate A and $B\}$. Then, **domain** $(R) = \{(a: (a, b) \in R\}$, **range** $(R) = \{b: (a, b) \in R\}$ and set B is called **codomain** of R.
- Let A and B be two sets and R be a relation from set A to set B.
  Then, the **inverse relation** from B to A is $R^{-1} = \{(b, a): (a, b) \in R\}$
- A relation f from a non-empty set A to a non-empty set B is a **function**, if every element of set A has one and only one image in set B. i.e. $f: A \to B$.
- Two functions f and g are said to be **equal** iff
  (i) domain of f = domain of g. (ii) codomain of f = codomain of g.
  Here, set A is **domain**, set B is **codomain** and subset of B containing the images of elements of A is range
- A function which has either R or one of its subsets as its range is called a **real valued function** and if its domain is also either R or a subset of R.
- **Algebra of Real Functions** Let $f: D_1 \to R$ and $g: D_2 \to R$ be two real functions with domain D_1 and D_2, respectively. Then,
  (i) $(f + g)(x) = f(x) + g(x), \forall x \in D_1 \cap D_2$. (ii) $(f - g)(x) = f(x) - g(x), \forall x \in D_1 \cap D_2$.
  (iii) $(f\,g)(x) = f(x) \cdot g(x), \forall x \in D_1 \cap D_2$. (iv) $\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \forall x \in D_1 \cap D_2 - \{x: g(x) \neq 0\}$.
  (v) $(cf)(x) = c \cdot f(x), \forall x \in D_1$, where c is scalar (real number).
- **Different Types of Real Functions** Suppose f is a function defined from real to real i.e. $f: R \to R$. Then function defined as

  (i) $f(x) = x, \ \forall\ x \in R$ is said to be an identity function.
  (ii) $f(x) = k, \ \forall\ x \in R$ is said to be a constant function.
  (iii) $f(x) = |x|, \ \forall\ x \in R$ is said to be a absolute function.
  (iv) $f(x) = \begin{cases} \frac{|x|}{x}, \text{ when } x \neq 0 \\ 0, \text{ when } x = 0 \end{cases}$ is said to be a signum.
  (v) $f(x) = [x], \ \forall\ x \in R$ is said to be a greatest integer function.
  (vi) $f(x) = \sqrt{x}, \ \forall\ x \in R$ is said to be a square root function.
  (vii) $f(x) = \frac{1}{x}, \ \forall\ x \in R$ is said to be a reciprocal function.
  (viii) $f(x) = a^x, \ \forall\ x \in R$ is said to be a exponential function,
  (ix) $f(x) = \log_a x, a > 0, a \neq 1, x > 0$ is said to be a logarithmic function.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1. If two sets A and B are having 99 elements in common, then the number of elements common to each of the sets $A \times B$ and $B \times A$ are

(a) 2^{99} (b) 99^2

(c) 100 (d) 18

2. If $A = \{1, 2, 5, 6\}$ and $B = \{1, 2, 3\}$, then what is $(A \times B) \cap (B \times A)$ equal to?

(a) $\{(1, 1), (2, 1), (6, 1), (3, 2)\}$

(b) $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$

(c) $\{(1, 1), (2, 2)\}$

(d) $\{(1, 1), (1, 2), (2, 5), (2, 6)\}$

3. Let $R = \{x \mid x \in N, x \text{ is a multiple of 3 and } x \le 100\}$
$S = \{x \mid x \in N, x \text{ is a multiple of 5 and } x \le 100\}$.
What is the number of elements in $(R \times S) \cap (S \times R)$?

(a) 36 (b) 33

(c) 20 (d) 6

4. If A is a finite set having n elements, then the number of relations which can be defined in A is

(a) 2^n (b) n^2

(c) 2^{n^2} (d) n^n

5. Let R be a relation in N defined by

$$R = \{(1 + x, 1 + x^2) : x \le 5, x \in N\}.$$

Which of the following is false?

(a) $R = \{(2, 2), (3, 5), (4, 10), (5, 17), (6, 25)\}$

(b) Domain of $R = \{2, 3, 4, 5, 6\}$

(c) Range of $R = \{2, 5, 10, 17, 26\}$

(d) None of the above

6. If $f(x) = \dfrac{1+x}{1-x}$, then $\dfrac{f(x) \cdot f(x^2)}{1 + [f(x)]^2}$ is equal to

(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{6}$

(c) $\dfrac{1}{8}$ (d) $\dfrac{1}{2}$

7. Let $f : R \to R$ be such that $f(x) = 2^x$. Then, consider the following statements

I. Range of $f = (0, \infty)$ II. $\{x : f(x) = 1\} = \{0\}$

III. $f(x + y) = f(x) \cdot f(y)$

Which of the above statements is/are correct?

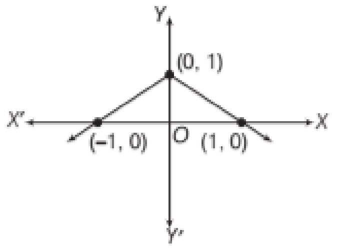
(a) Only I (b) I and II

(c) II and III (d) All of these

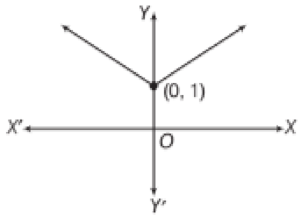
8. The graph of the function defined by

$$f(x) = \begin{cases} 1 - x, & x < 0 \\ 1, & x = 0 \\ 1 + x, & x > 0 \end{cases} \text{ is}$$

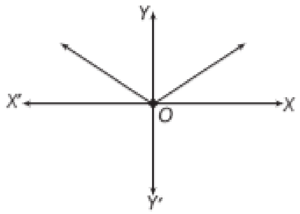
(a)



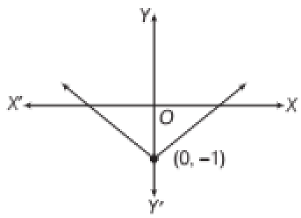
(b)



(c)



(d)



9. If $f(x) = \cos(\log x)$, then

$f(x)\,f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value

(a) -1 (b) $\frac{1}{2}$ (c) -2 (d) 0

10. If a function f satisfies $f\{f(x)\} = x + 1$ for all real values of x and if $f(0) = \frac{1}{2}$, then $f(1)$ is equal to

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{3}{2}$ (d) 2

VERY SHORT ANSWER Type Questions

11. If $\left(\frac{x}{3} + 1, y - \frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$, then find the values of x and y. **[NCERT]**

12. If $n(A) = m$ and $n(B) = n$, then find the total number of non-empty relations that can be defined from A to B. **[NCERT Exemplar]**

13. If $A \times B = \{(1, x), (1, y), (1, z), (1, w)\}$, then find A and B. **[NCERT]**

14. If $A \times B = \{(a, 1), (a, 5), (a, 2), (b, 2), (b, 5), (b, 1)\}$, then find A, B and $B \times A$.

15. If the set A has 3 elements and set $B = \{3, 4, 5\}$, then find the number of elements in $A \times B$.

16. If $A = \{a, b\}, B = \{2, 3, 5, 6, 7\}$ and $C = \{5, 6, 7, 8, 9\}$, then find $A \times (B \cap C)$.

17. If $A = \{-1, 1\}$, then find $A \times A \times A$.

18. If $A = \{1, 2\}$ and $B = \{3, 4, 5\}$. Then, write down
- (i) cartesian product $A \times B$.
- (ii) the relation $R : A \to B$ such that $(a + b)$ is an even number.

19. If $A = \{0, 1, 2\}$ and $B = \{1, 2, 3, 4, 5\}$, then represent the rule $f: A \to B$ given by $f(x) = x + 2$ by an arrow diagram.

20. If $X = \{0, \pm 2, 4\}$ and $Y = \{0, 4, 5, 16\}$, then represent the rule $f: X \to Y$ given by $f(x) = x^2$ by an arrow diagram.

21. If $A = \{9, 10, 11, 12, 13\}$ and $f : A \to N$ be defined by $f(n) =$ the highest prime factor of n, then find the range of f.

22. If $f(x) = [x]$, then find $(3f)(x)$.

23. Find the range of each of the following function. $f(x) = x$, x is a real number. **[NCERT]**

SHORT ANWER Type I Questions

24. If $U = \{1, 2, 3, 4\}$ and $R = \{(x, y) : y > x$ for all, $x, y \in U\}$, then find domain and range of R.

25. Find the domain and range of the relation $R = \{(x, y) : x + y = 8 ; x, y \in N\}$

26. If $N = \{1, 2, 3\}$, then find the relation $R = \{(x, y) : x \in N, y \in N$ and $2x + y = 10\}$ in $N \times N$.
Also, find its domain and range.

27. Which of the following are functions, if $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4, 5\}$?
- (i) $f_1 = \{(a, 1), (b, 1), (c, 3), (d, 4)\}$
- (ii) $f_2 = \{(a, 1), (b, 2), (c, 4), (a, 2), (d, 5)\}$

28. Write the range of $y = \frac{|x - 1|}{x - 1}$.

29. Find the range of $f(x) = 2 - 3x$, $x \in R$, $x > 0$.

30. Find the range of $f(x) = x^2 + 2$, $x \in R$.

31. If $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$, then find $f(1 + h)$.

32. If $f(x) = \frac{x + 1}{x - 1}$, then find $f(x^2)$ and $[f(x)]^2$.

SHORT ANSWER Type II Questions

33. Let A and B be two sets, such that $A \times B$ consists of 6 elements. If 3 elements of $A \times B$ are (1, 4), (2, 6) and (3, 6), then find $A \times B$ and $B \times A$.

34. Determine the domain and range of the relation R, where $R = \{(2x + 3, x^3) : x$ is a prime number less than $10\}$.

35. If $X = \{1, 2, 3, 4, 5\}$, $Y = \{1, 2, 5, 6, 7, 9, 10, 11, 12, 13, 14\}$ and $f : X \to Y$ be defined by $f(x) = 2x + 3$, then find the domain and range of f.

36. Is $g = \{(2, 3), (4, 5), (6, 7)\}$ a function? If this is described by the formula $g(x) = \alpha x + 2\beta$, then what values would be assigned to α and β ?

37. If A and B be two sets, such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1), (y, 2)$ and $(z, 1)$ are in $A \times B$, then find A, B and $A \times B$, where x, y and z are distinct elements. **[NCERT]**

38. Find the domain of the function f defined by $f(x) = \sqrt{4 - x} + \frac{1}{\sqrt{x^2 - 1}}$. **[NCERT Exemplar]**

39. If a function $f : R \to R$ be defined by

$$f(x) = \begin{cases} 3x - 2, & x < 0 \\ 1, & x = 0 \\ 4x + 1, & x > 0 \end{cases}$$

Find $f(1)$, $f(-1)$, $f(0)$, $f(2)$.

40. If $f(x) = x^2$, find $\frac{f(1.1) - f(1)}{1.1 - 1}$ [NCERT]

41. If $f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \le x < 1 \\ \frac{1}{x}, & x \ge 1 \end{cases}$

Find (i) $f\left(\frac{1}{2}\right)$ (ii) $f(-2)$ (iii) $f(1)$ (iv) $f(\sqrt{3})$

42. If $f(x) = x^3 - \frac{1}{x^3}$, show that $f(x) + f\left(\frac{1}{x}\right) = 0$.

43. If $A = \{1, 2\}, B = \{1, 2, 3, 4\}, C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$, then verify that [NCERT]

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(ii) $A \times C$ is a subset of $B \times D$.

44. If $A = \{1, 2, 3\}, B = \{2, 3, 4\}, C = \{1, 3, 4\}$ and $D = \{2, 4, 5\}$, then verify that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

45. Let $A = \{1, 2, 3, ..., 14\}$. Define a relation R from A to A by $R = \{(x, y) : 3x - y = 0$, where $x, y \in A\}$. Depict this relationship using an arrow diagram. Write down its domain, codomain and range. [NCERT]

LONG ANSWER Type Questions

46. Find the domain and range of the following relations.

(i) $R = \{(x, y) : x, y \in N, y = x^2 + 3 \text{ and } 0 < x < 5\}$

(ii) $R = \{(x, y) : x, y \in N, y = \frac{1}{1+x}$ and x is odd natural number$\}$

47. Determine the domain and range of the following relations.

(i) $R_1 = \left\{\left(x, \frac{1}{x}\right) : 0 < x < 6, x \in N\right\}$

(ii) $R_2 = \{(x, x^2) : x \text{ is prime number less than } 10\}$

48. If $f(x) = \begin{cases} 1 + x, & -1 \le x < 0 \\ x^2 - 1, & 0 < x < 2 \\ 2x, & 2 \le x \end{cases}$

Then, find $f(3)$, $f(-2)$, $f(0)$, $f\left(\frac{1}{2}\right)$, $f(2 - h)$ and $f(-1 + h)$, where $h > 0$ is very small.

49. Write the following after removing the modulus $f(x) = |2x - 1|, -1 \le x \le 1$.

50. Let $A = \{-3, -2, -1, 4\}$ and $f : A \to Z$ given by $f(x) = x^2 + x + 2$. Find

(i) the range of f. (ii) pre-images of 6 and 4.

51. Find the domain and range of the following real functions. [NCERT]

(i) $f(x) = -|2x|$ (ii) $f(x) = \sqrt{9 - x^2}$

52. Find the range of the function $f(x) = 1 - |x + 2|$.

53. Find the range of the function $f(x) = \frac{|x + 4|}{x + 4}$.

54. (i) Let $A = \{8, 11, 12, 15, 18, 23\}$ and f is a function from $A \to N$ such that $f(x) =$ highest prime factor of x, find f and its range.

(ii) Find the domain of the function $f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$.

55. (i) If $f(x) = \sqrt{x^2 + 1}$ and $g(x) = 2x^2 + 3$, then find $(f + g)(x)$.

(ii) Find the product of the identity function and the modulus function.

56. The relation f is defined by $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$ and the relation g is defined by $g(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 3x, & 2 \le x \le 10 \end{cases}$.

Show that f is a function and g is not a function.

57. Find the simplified form of $f(x) = |x - 2| + |2 + x|$, if $-3 \le x \le 3$. [NCERT Exemplar]

58. Everyone wants to be a perfect ideal human being. Let us assume that dishonesty is one of the factors that affects our perfectness and perfectness has an inverse relation with dishonesty. For any value x of level of dishonesty, we have a unique value y of perfection.

(i) Write down the equation that relates y with x.

(ii) Does this relationship from $x \in (0, \infty)$ to $y \in (0, \infty)$, form a function?

(iii) For what level of dishonesty one can achieve $\left(\frac{1}{4}\right)$th level of perfection?

(iv) What will be the change in level of perfection when the level of dishonesty changes from 4 to 2?

59. (i) Find the domain and range of the function $f(x)=\frac{x^2-9}{x-3}$.

(ii) Find the domain of the function $f(x)=\frac{x^2+3x+5}{x^2+x-6}$.

60. Find domain and range of the real function f(x) defined by

$$f(x)=\begin{cases}1-x, & x<0\\ 1, & x=0\\ x-1, & x>0\end{cases}$$ and draw its graph.

61. State whether each of the following statements are true or false. If the statement is false, then rewrite the given statement correctly.

(i) If $P=\{m,n\}$ and $Q=\{n,m\}$, then $P\times Q=\{(m,n),(n,m)\}$. [NCERT]

(ii) If A and B are non-empty sets, then $A\times B$ is a non-empty set of ordered pairs (x,y) such that $x\in B$ and $y\in A$.

(iii) If $A=\{1,2\}$ and $B=\{3,4\}$, then $A\times(B\cap\phi)=\phi$.

HINTS & ANSWERS

1. (b) $n[(A\times B)\cap(B\times A)]=n[(A\cap B)\times(B\cap A)]$
$=n(A\cap B)\times n(B\cap A)=99\times 99=99^2$

2. (b) Given, $A=\{1,2,5,6\}$ and $B=\{1,2,3\}$

$A\times B=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(5,1),(5,2),(5,3),(6,1),(6,2),(6,3)]$

$B\times A=\{(1,1),(1,2),(1,5),(1,6),(2,1),(2,2),(2,5),(2,6),(3,1),(3,2),(3,5),(3,6)]$.

$\therefore\ (A\times B)\cap(B\times A)=\{(1,1),(1,2),(2,1),(2,2)\}$

3. (a) $\because\ R=\{3,6,9,12,15,\ldots,99\}$

and $S=\{5,10,15,\ldots,95,100\}$

Now, $(R\times S)\cap(S\times R)=(R\cap S)\times(S\cap R)$
$=\{15,30,45,60,75,90\}\times\{15,30,45,60,75,90\}$

$\therefore$ Number of elements in
$(R\times S)\cap(S\times R)=6\times 6=36$

4. (c) Number of relations from A to A is $2^{n\times n}$.

5. (a) $R=\{(2,2),(3,5),(4,10),(5,17),(6,26)\}$

Domain of $R=\{x:(x,y)\in R\}=\{2,3,4,5,6\}$

and range of $R=\{y:(x,y)\in R\}=\{2,5,10,17,26\}$

$\therefore$ Hence option (a) is false.

6. (d) $f(x^2)=\frac{1+x^2}{1-x^2}=\frac{(1+x^2)}{(1+x)(1-x)}$

Now, $\frac{f(x)\cdot f(x^2)}{1+[f(x)]^2}$

$$=\frac{\frac{(1+x)}{(1-x)}\times\frac{(1+x^2)}{(1+x)(1-x)}}{1+\frac{(1+x)^2}{(1-x)^2}}=\frac{\frac{(1+x^2)}{(1-x)^2}}{\frac{(1-x)^2+(1+x)^2}{(1-x)^2}}$$

$$=\frac{1+x^2}{2+2x^2}=\frac{(1+x^2)}{2(1+x^2)}=\frac{1}{2}$$

7. (d) I. Since, 2^x is positive for every $x\in R$. So, $f(x)=2^x$ is a positive real number for every $x\in R$.

Moreover, for every positive real number x, there exist $\log_2 x\in R$ such that $f(\log_2 x)=2^{\log_2 x}=x$

Hence, we conclude that the range of f is the set of all positive real numbers.

II. $f(x)=1\Rightarrow 2^x=1\Rightarrow 2^x=2^0\Rightarrow x=0$

$\therefore\ \{x:f(x)=1\}=\{0\}$

III. We have, $f(x)=2^x$

$\therefore\ f(x+y)=2^{x+y}=2^x\cdot 2^y=f(x)\cdot f(y)$

Thus, $f(x+y)=f(x)\cdot f(y)$ holds for all $x,y\in R$.

Hence, all the statements are correct.

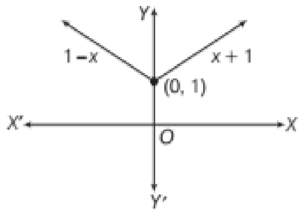
8. (b) We have, $f(x)=\begin{cases}1-x, & x<0\\ 1, & x=0\\ x+1, & x>0\end{cases}$

Let $f(x)=y$. Then, $y=1-x$ or $x+y=1,\ \forall\ x<0$

$y=1$ for $x=1$

$y=x+1\Rightarrow -x+y=1,\ \forall\ x>0$

Now, graph of $f(x)$ is as shown below



9. (d) We have, $f(x)=\cos(\log x)$

$\therefore\ f(x)\,f(y)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]$

$=\cos(\log x)\cos(\log y)-\frac{1}{2}[2\cos(\log x)\cos(\log y)]$

$=\cos(\log x)\cos(\log y)-\cos(\log x)\cos(\log y)=0$

10. (c) We have, $f\{f(x)\}=x+1,\ \forall\ x\in R$ ...(i)

$\therefore\ f\{f(0)\}=0+1=1$

$\Rightarrow \quad f\left(\frac{1}{2}\right) = 1 \qquad [\because f(0) = 1/2] \; ...(ii)$

Now, take $x = \frac{1}{2}$ in Eq. (i), then we get

$$f\left\{f\left(\frac{1}{2}\right)\right\} = \frac{1}{2} + 1$$

$\Rightarrow \quad f(1) = \frac{3}{2} \qquad$ [using Eq. (ii)]

11. $x = 2,\ y = 1$

12. $2^{mn} - 1$

13. $A = \{1\},\ B = \{x, y, z, w\}$

14. $A = \{a, b\},\ B = \{1, 5, 2\}$

$B \times A = \{(1, a), (1, b), (5, a), (5, b)(2, a)(2, b)\}$

15. 9

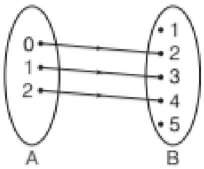
16. $\{(a, 5), (a, 6), (a, 7), (b, 5), (b, 6), (b, 7)\}$

17. $\{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1)(1, 1, 1)\}$

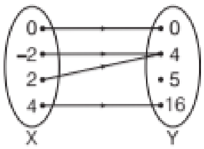
18. (i) $A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$

(ii) $R : A \to B$ is $R = \{(1, 3), (1, 5), (2, 4)\}$

19.



20.



21. $\{3, 5, 11, 13\}$

22. $3[x]$

23. Range $= R$

24. Domain $(R) = \{1, 2, 3, 4\}$, Range $(R) = \{2, 3, 4\}$

25. Domain $(R) = \{1, 2, 3, 4, 5, 6, 7\}$

Range $(R) = \{7, 6, 5, 4, 3, 2, 1\}$

26. $R = \{(1, 8), (2, 6), (3, 4)\}$

Domain $(R) = \{1, 2, 3\}$ and Range $(R) = \{8, 6, 4\}$

27. (i) f_1 is a function.

(ii) $\because f(a) = 1$ and $f(a) = 2$, f_2 is not a function.

28. $\{-1, 1\}$

29. $(-\infty, 2)$

30. $(2, \infty)$

31. $\dfrac{h^2 + h + 1}{h^2 + 3h + 3}$

32. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}, (f(x))^2 = \left(\dfrac{x + 1}{x - 1}\right)^2$

33. $A \times B = \{(1, 4), (1, 6), (2, 4), (2, 6), (3, 4), (3, 6)\}$

$B \times A = \{(4, 1), (4, 2), (4, 3), (6, 1), (6, 2), (6, 3)\}$

34. Domain $= \{7, 9, 13, 17\}$ Range $= \{8, 27, 125, 343\}$

35. Domain $= \{1, 2, 3, 4, 5\}$ Range $= \{5, 7, 9, 11, 13\}$

36. g is a function; $\alpha = 2,\ \beta = -1$

37. $A = \{x, y, z\},\ B = \{1, 2\}$

$A \times B = \{(x, 1), (x, 2), (y, 1), (y, 2), (z, 1), (z, 2)\}$

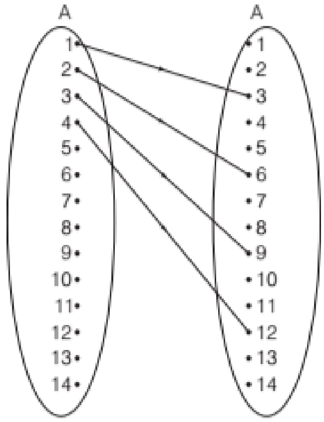
38. Domain $= (-\infty, -1) \cup (1, 4]$

39. $f(1) = 5,\ f(-1) = -5,\ f(0) = 1,\ f(2) = 9$

40. 2.1

41. (i) $\frac{1}{2}$ (ii) 4 (iii) 1 (iv) $\frac{1}{\sqrt{3}}$

45.



Domain $= \{1, 2, 3, 4\}$, Codomain $= A$,
Range $= \{3, 6, 9, 12\}$

46. (i) Domain $= \{1, 2, 3, 4\}$ Range $= \{4, 7, 12, 19\}$

(ii) Domain $= \{1, 3, 5, ...\}$ Range $= \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ...\right\}$

47. (i) Domain $= \{1, 2, 3, 4, 5\}$ Range $= \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$

(ii) Domain $= \{2, 3, 5, 7\}$ Range $= \{4, 9, 25, 49\}$

48. $f(3) = 6,\ f(-2) =$ Not defined, $f(0) =$ Not defined,

$f\left(\frac{1}{2}\right) = \frac{-3}{4};\ f(2 - h) = h^2 - 4h + 3,\ f(-1 + h) = h$

49. $f(x) = \begin{cases} 1 - 2x, & -1 \le x \le \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \le x \le 1 \end{cases}$

50. (i) Range $(f) = \{8, 4, 2, 22\}$

(ii) There is no pre-image of 6, pre-image of 4 is -2.

51. (i) Domain $= R$ Range $= (-\infty, 0]$

(ii) Domain $= [-3, 3]$ Range $= [0, 3]$

52. $(-\infty, 1]$

53. $\{-1, 1\}$

54. (i) $f = \{(8, 2), (11, 11), (12, 3), (15, 5), (18, 3), (23, 23)\}$
Range $= \{2, 3, 5, 11, 23\}$

(ii) Domain $= R - \{2, 6\}$

55. (i) $(f + g)(x) = \sqrt{x^2 + 1} + 2x^2 + 3$

(ii) $(f \cdot g)(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

57. $f(x) = \begin{cases} 2x & 2 \leq x \leq 3 \\ 4, & -2 \leq x < 2 \\ -2x, & -3 \leq x < -2 \end{cases}$

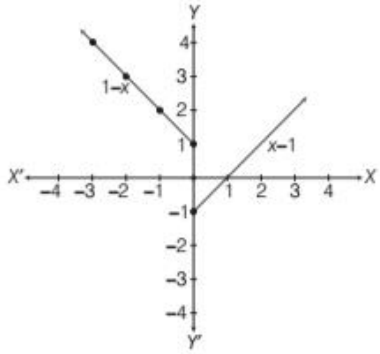
58. (i) $y = \frac{k}{x}$, k is constant (ii) Yes

(iii) 4^{th} level (iv) $\frac{1}{4}$ to $\frac{1}{2}$

59. (i) Domain $= R - \{3\}$
Range $= R - \{6\}$

(ii) Domain $= R - \{-3, 2\}$

60. Domain $= R$ Range $= (-1, \infty)$



61. (i) It is false, because if $P = \{m, n\}$ and $Q = \{n, m\}$, then
$P \times Q = \{m, n\} \times \{n, m\} = \{(m, n), (m, m), (n, n), (n, m)\}$

(ii) It is false, correct statement is "If A and B are non-empty sets, then $A \times B$ is a non-empty set of ordered pairs (x, y), such that $x \in A$ and $y \in B$".

(iii) It is true, because $B \cap \phi = \phi$

$\therefore \quad A \times (B \cap \phi) = A \times \phi = \phi$